5.1 Simplifying Algebraic Fractions

5.1 OBJECTIVES

1. Find the GCF for two monomials and simplify a fraction
2. Find the GCF for two polynomials and simplify a fraction

Much of our work with algebraic fractions will be similar to your work in arithmetic. For instance, in algebra, as in arithmetic, many fractions name the same number. You will remember from Chapter 0 that

$$\frac{1}{4} = \frac{1 \cdot 2}{4 \cdot 2} = \frac{2}{8}$$

or

$$\frac{1}{4} = \frac{1 \cdot 3}{4 \cdot 3} = \frac{3}{12}$$

So $\frac{1}{4}, \frac{2}{8}$, and $\frac{3}{12}$ all name the same number. They are called **equivalent fractions.** These examples illustrate what is called the **Fundamental Principle of Fractions.** In algebra it becomes

Rules and Properties: Fundamental Principle of Algebraic Fractions

For polynomials P, Q, and R,

$$\frac{P}{Q} = \frac{PR}{QR} \qquad \text{when } Q \neq 0 \text{ and } R \neq 0$$

This principle allows us to multiply or divide the numerator and denominator of a fraction by the same nonzero polynomial. The result will be an expression that is equivalent to the original one.

Our objective in this section is to simplify algebraic fractions by using the fundamental principle. In algebra, as in arithmetic, to write a fraction in simplest form, you divide the numerator and denominator of the fraction by their greatest common factor (GCF). The numerator and denominator of the resulting fraction will have no common factors other than 1, and the fraction is then in **simplest form.** The following rule summarizes this procedure.

NOTE Notice that step 2 uses the Fundamental Principle of Fractions. The GCF is R in the rule above.

Step by Step: To Write Algebraic Fractions in Simplest Form

Step 1 Factor the numerator and denominator.

Step 2 Divide the numerator and denominator by the greatest common factor (GCF). The resulting fraction will be in lowest terms.

Example 1

Writing Fractions in Simplest Form

NOTE This is the same as dividing both the numerator and denominator of $\frac{18}{30}$ by 6.

(a) Write $\frac{18}{30}$ in simplest form.

$$\frac{18}{30} = \frac{2 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 5} = \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{3}} \cdot 3}{\underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{3}} \cdot 5} = \frac{3}{5}$$

Divide by the GCF. The slash lines indicate that we have divided the numerator and denominator by 2 and by 3.

(b) Write $\frac{4x^3}{6x}$ in simplest form.

$$\frac{4x^3}{6x} = \frac{\overset{1}{\cancel{2}} \cdot 2 \cdot \overset{1}{\cancel{x}} \cdot x \cdot x}{\underset{1}{\cancel{2}} \cdot 3 \cdot \underset{1}{\cancel{x}}} = \frac{2x^2}{3}$$

(c) Write $\frac{15x^3y^2}{20xy^4}$ in simplest form.

$$\frac{15x^3y^2}{20xy^4} = \frac{3 \cdot \overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{x}} \cdot x \cdot x \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{y}}}{2 \cdot 2 \cdot \underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{y}} \cdot y \cdot y} = \frac{3x^2}{4y^2}$$

(d) Write $\frac{3a^2b}{9a^3b^2}$ in simplest form.

$$\frac{3a^2b}{9a^3b^2} = \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{b}}}{\underset{1}{\cancel{3}} \cdot 3 \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{a}} \cdot a \cdot \underset{1}{\cancel{b}} \cdot b} = \frac{1}{3ab}$$

(e) Write $\frac{10a^5b^4}{2a^2b^3}$ in simplest form.

$$\frac{10a^5b^4}{2a^2b^3} = \frac{5 \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot a \cdot a \cdot a \cdot \overset{1}{\cancel{b}} \cdot \overset{1}{\cancel{b}} \cdot \overset{1}{\cancel{b}} \cdot b}{\underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{b}}} = \frac{5a^3b}{1} = 5a^3b$$

NOTE Most of the methods of this chapter build on our factoring work of the last chapter.

CHECK YOURSELF 1

Write each fraction in simplest form.

(a) $\frac{30}{66}$ **(b)** $\frac{5x^4}{15x}$ **(c)** $\frac{12xy^4}{18x^3y^2}$ **(d)** $\frac{5m^2n}{10m^3n^3}$ **(e)** $\frac{12a^4b^6}{2a^3b^4}$

In simplifying arithmetic fractions, common factors are generally easy to recognize. With algebraic fractions, the factoring techniques you studied in Chapter 4 will have to be used as the *first step* in determining those factors.

Example 2

Writing Fractions in Simplest Form

Write each fraction in simplest form.

(a) $$\frac{2x - 4}{x^2 - 4} = \frac{2(x - 2)}{(x + 2)(x - 2)}$$ Factor the numerator and denominator.

$$= \frac{2\overset{1}{\cancel{(x - 2)}}}{(x + 2)\underset{1}{\cancel{(x - 2)}}}$$ Divide by the GCF $x - 2$. The slash lines indicate that we have divided by that common factor.

$$= \frac{2}{x + 2}$$

(b) $$\frac{3x^2 - 3}{x^2 - 2x - 3} = \frac{3(x - 1)\overset{1}{\cancel{(x + 1)}}}{(x - 3)\underset{1}{\cancel{(x + 1)}}}$$

$$= \frac{3(x - 1)}{x - 3}$$

(c) $$\frac{2x^2 + x - 6}{2x^2 - x - 3} = \frac{(x + 2)\overset{1}{\cancel{(2x - 3)}}}{(x + 1)\underset{1}{\cancel{(2x - 3)}}}$$

$$= \frac{x + 2}{x + 1}$$

CAUTION

Pick any value, other than 0, for x and substitute. You will quickly see that

$$\frac{x + 2}{x + 1} \neq \frac{2}{1}$$

Be Careful! The expression $\frac{x + 2}{x + 1}$ is already in simplest form. Students are often tempted to divide as follows:

$$\frac{\cancel{x} + 2}{\cancel{x} + 1} \quad \text{is } \textit{not equal} \text{ to} \quad \frac{2}{1}$$

The x's are *terms* in the numerator and denominator. They *cannot* be divided out. Only *factors* can be divided. The fraction

$$\frac{x + 2}{x + 1}$$

is in its simplest form.

CHECK YOURSELF 2

Write each fraction in simplest form.

(a) $\frac{5x - 15}{x^2 - 9}$ **(b)** $\frac{a^2 - 5a + 6}{3a^2 - 6a}$

(c) $\frac{3x^2 + 14x - 5}{3x^2 + 2x - 1}$ **(d)** $\frac{5p - 15}{p^2 - 4}$

Remember the rules for signs in division. The quotient of a positive number and a negative number is always negative. Thus there are three equivalent ways to write such a quotient. For instance,

$$\frac{-2}{3} = \frac{2}{-3} = -\frac{2}{3}$$

NOTE $\frac{-2}{3}$, with the negative sign in the numerator, is the most common way to write the quotient.

The quotient of two positive numbers or two negative numbers is always positive. For example,

$$\frac{-2}{-3} = \frac{2}{3}$$

Example 3

Writing Fractions in Simplest Form

Write each fraction in simplest form.

NOTE In part (a), the final quotient is written in the most common way with the minus sign in the numerator.

(a) $$\frac{6x^2}{-3xy} = \frac{2 \cdot \overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{x}} \cdot x}{(-1) \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{x}} \cdot y} = \frac{2x}{-y} = \frac{-2x}{y}$$

(b) $$\frac{-5a^2b}{-10b^2} = \frac{\overset{1}{\cancel{(-1)}} \cdot \overset{1}{\cancel{5}} \cdot a \cdot a \cdot \overset{1}{\cancel{b}}}{\underset{1}{\cancel{(-1)}} \cdot 2 \cdot \underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{b}} \cdot b} = \frac{a^2}{2b}$$

CHECK YOURSELF 3

Write each fraction in simplest form.

(a) $\frac{8x^3y}{-4xy^2}$ (b) $\frac{-16a^4b^2}{-12a^2b^5}$

It is sometimes necessary to factor out a monomial before simplifying the fraction.

Writing Fractions in Simplest Form

Write each fraction in simplest form.

(a) $$\frac{6x^2 + 2x}{2x^2 + 12x} = \frac{2x(3x + 1)}{2x(x + 6)} = \frac{3x + 1}{x + 6}$$

(b) $$\frac{x^2 - 4}{x^2 + 6x + 8} = \frac{(x + 2)(x - 2)}{(x + 2)(x + 4)} = \frac{x - 2}{x + 4}$$

CHECK YOURSELF 4

Simplify each fraction.

(a) $\dfrac{3x^3 - 6x^2}{9x^4 - 3x^2}$ **(b)** $\dfrac{x^2 - 9}{x^2 - 12x + 27}$

Reducing certain algebraic fractions will be easier with the following result. First, verify for yourself that

$$5 - 8 = -(8 - 5)$$

In general, it is true that

$$a - b = -(b - a)$$

or, by dividing both sides of the equation by $b - a$,

$$\frac{a - b}{b - a} = \frac{-(b - a)}{b - a}$$

So dividing by $b - a$ on the right, we have

NOTE Remember that *a* and *b* cannot be divided out because they are not factors.

$$\frac{a - b}{b - a} = -1$$

Let's look at some applications of that result in Example 5.

Writing Fractions in Simplest Form

Write each fraction in simplest form.

(a) $\dfrac{2x - 4}{4 - x^2} = \dfrac{2(x - 2)}{(2 + x)(2 - x)}$ This is equal to -1.

$$= \frac{2(-1)}{2 + x} = \frac{-2}{2 + x}$$

(b) $\dfrac{9 - x^2}{x^2 + 2x - 15} = \dfrac{(3 + x)(3 - x)}{(x + 5)(x - 3)}$ This is equal to -1.

$$= \frac{(3 + x)(-1)}{x + 5}$$

$$= \frac{-x - 3}{x + 5}$$

CHECK YOURSELF 5

Write each fraction in simplest form.

(a) $\dfrac{3x - 9}{9 - x^2}$

(b) $\dfrac{x^2 - 6x - 27}{81 - x^2}$

CHECK YOURSELF ANSWERS

1. (a) $\dfrac{5}{11}$; (b) $\dfrac{x^3}{3}$; (c) $\dfrac{2y^2}{3x^2}$; (d) $\dfrac{1}{2mn^2}$; (e) $6ab^2$
2. (a) $\dfrac{5}{x + 3}$; (b) $\dfrac{a - 3}{3a}$; (c) $\dfrac{x + 5}{x + 1}$; (d) $\dfrac{5(p - 3)}{(p + 2)(p - 2)}$
3. (a) $\dfrac{-2x^2}{y}$; (b) $\dfrac{4a^2}{3b^3}$
4. (a) $\dfrac{x - 2}{3x^2 - 1}$; (b) $\dfrac{x + 3}{x - 9}$
5. (a) $\dfrac{-3}{x + 3}$; (b) $\dfrac{-x - 3}{x + 9}$

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5.1 Exercises

Write each fraction in simplest form.

1. $\frac{16}{24}$

2. $\frac{56}{64}$

3. $\frac{80}{180}$

4. $\frac{18}{30}$

5. $\frac{4x^5}{6x^2}$

6. $\frac{10x^2}{15x^4}$

7. $\frac{9x^3}{27x^6}$

8. $\frac{25w^6}{20w^2}$

9. $\frac{10a^2b^5}{25ab^2}$

10. $\frac{18x^4y^3}{24x^2y^3}$

11. $\frac{42x^3y}{14xy^3}$

12. $\frac{18pq}{45p^2q^2}$

13. $\frac{2xyw^2}{6x^2y^3w^3}$

14. $\frac{3c^2d^2}{6bc^3d^3}$

15. $\frac{10x^5y^5}{2x^3y^4}$

16. $\frac{3bc^6d^3}{bc^3d}$

17. $\frac{-4m^3n}{6mn^2}$

18. $\frac{-15x^3y^3}{-20xy^4}$

19. $\frac{-8ab^3}{-16a^3b}$

20. $\frac{14x^2y}{-21xy^4}$

ANSWERS

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ANSWERS

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42. ______

21. $\dfrac{8r^2s^3t}{-16rs^4t^3}$

22. $\dfrac{-10a^3b^2c^3}{15ab^4c}$

23. $\dfrac{3x + 18}{5x + 30}$

24. $\dfrac{4x - 28}{5x - 35}$

25. $\dfrac{3x - 6}{5x - 15}$

26. $\dfrac{x^2 - 25}{3x - 15}$

27. $\dfrac{6a - 24}{a^2 - 16}$

28. $\dfrac{5x - 5}{x^2 - 4}$

29. $\dfrac{x^2 + 3x + 2}{5x + 10}$

30. $\dfrac{4w^2 - 20w}{w^2 - 2w - 15}$

31. $\dfrac{x^2 - 6x - 16}{x^2 - 64}$

32. $\dfrac{y^2 - 25}{y^2 - y - 20}$

33. $\dfrac{2m^2 + 3m - 5}{2m^2 + 11m + 15}$

34. $\dfrac{6x^2 - x - 2}{3x^2 - 5x + 2}$

35. $\dfrac{p^2 + 2pq - 15q^2}{p^2 - 25q^2}$

36. $\dfrac{4r^2 - 25s^2}{2r^2 + 3rs - 20s^2}$

37. $\dfrac{2x - 10}{25 - x^2}$

38. $\dfrac{3a - 12}{16 - a^2}$

39. $\dfrac{25 - a^2}{a^2 + a - 30}$

40. $\dfrac{2x^2 - 7x + 3}{9 - x^2}$

41. $\dfrac{x^2 + xy - 6y^2}{4y^2 - x^2}$

42. $\dfrac{16z^2 - w^2}{2w^2 - 5wz - 12z^2}$

43. $\dfrac{x^2 + 4x + 4}{x + 2}$

44. $\dfrac{4x^2 + 12x + 9}{2x + 3}$

45. $\dfrac{xy - 2y + 4x - 8}{2y + 6 - xy - 3x}$

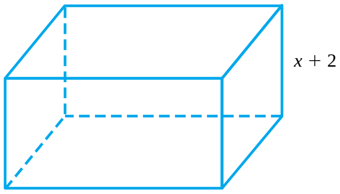
46. $\dfrac{ab - 3a + 5b - 15}{15 + 3a^2 - 5b - a^2b}$

47. $\dfrac{y - 7}{7 - y}$

48. $\dfrac{5 - y}{y - 5}$

49. The area of the rectangle is represented by $6x^2 + 19x + 10$. What is the length?

50. The volume of the box is represented by $(x^2 + 5x + 6)(x + 5)$. Find the polynomial that represents the area of the bottom of the box.



51. To work with algebraic fractions correctly, it is important to understand the difference between a *factor* and a *term* of an expression. In your own words, write difinitions for both, explaining the difference between the two.

52. Give some examples of terms and factors in algebraic fractions, and explain how both are affected when a fraction is reduced.

53. Show how the following algebraic fraction can be reduced:

$$\frac{x^2 - 9}{4x + 12}$$

Note that your reduced fraction is equivalent to the given fraction. Are there other algebraic fractions equivalent to this one? Write another algebraic fraction that you think is equivalent to this one. Exchange papers with another student. Do you agree that their fraction is equivalent to yours? Why or why not?

54. Explain the reasoning involved in each step of reducing the fraction $\dfrac{42}{56}$.

55. Describe why $\dfrac{3}{5}$ and $\dfrac{27}{45}$ are *equivalent fractions.*

ANSWERS

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52. ______
53. ______
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ANSWERS

a. ________

b. ________

c. ________

d. ________

e. ________

f. ________

g. ________

h. ________

Getting Ready for Section 5.2 [Section 0.2]

Perform the indicated operations.

(a) $\frac{3}{10} + \frac{4}{10}$ (b) $\frac{5}{8} - \frac{4}{8}$

(c) $\frac{5}{12} - \frac{1}{12}$ (d) $\frac{7}{16} + \frac{3}{16}$

(e) $\frac{7}{20} + \frac{9}{20}$ (f) $\frac{13}{8} - \frac{5}{8}$

(g) $\frac{11}{6} - \frac{2}{6}$ (h) $\frac{5}{9} + \frac{7}{9}$

Answers

1. $\frac{2}{3}$ **3.** $\frac{4}{9}$ **5.** $\frac{2x^3}{3}$ **7.** $\frac{1}{3x^3}$ **9.** $\frac{2ab^3}{5}$ **11.** $\frac{3x^2}{y^2}$ **13.** $\frac{1}{3xy^2w}$

15. $5x^2y$ **17.** $\frac{-2m^2}{3n}$ **19.** $\frac{b^2}{2a^2}$ **21.** $\frac{-r}{2st^2}$ **23.** $\frac{3}{5}$ **25.** $\frac{3(x-2)}{5(x-3)}$

27. $\frac{6}{a+4}$ **29.** $\frac{x+1}{5}$ **31.** $\frac{x+2}{x+8}$ **33.** $\frac{m-1}{m+3}$ **35.** $\frac{p-3q}{p-5q}$

37. $\frac{-2}{x+5}$ **39.** $\frac{-a-5}{a+6}$ **41.** $\frac{-x-3y}{2y+x}$ **43.** $x+2$ **45.** $\frac{-(y+4)}{y+3}$

47. -1 **49.** $2x+5$ **51.** **53.** **55.**

a. $\frac{7}{10}$ **b.** $\frac{1}{8}$ **c.** $\frac{1}{3}$ **d.** $\frac{5}{8}$ **e.** $\frac{4}{5}$ **f.** 1 **g.** $\frac{3}{2}$ **h.** $\frac{4}{3}$