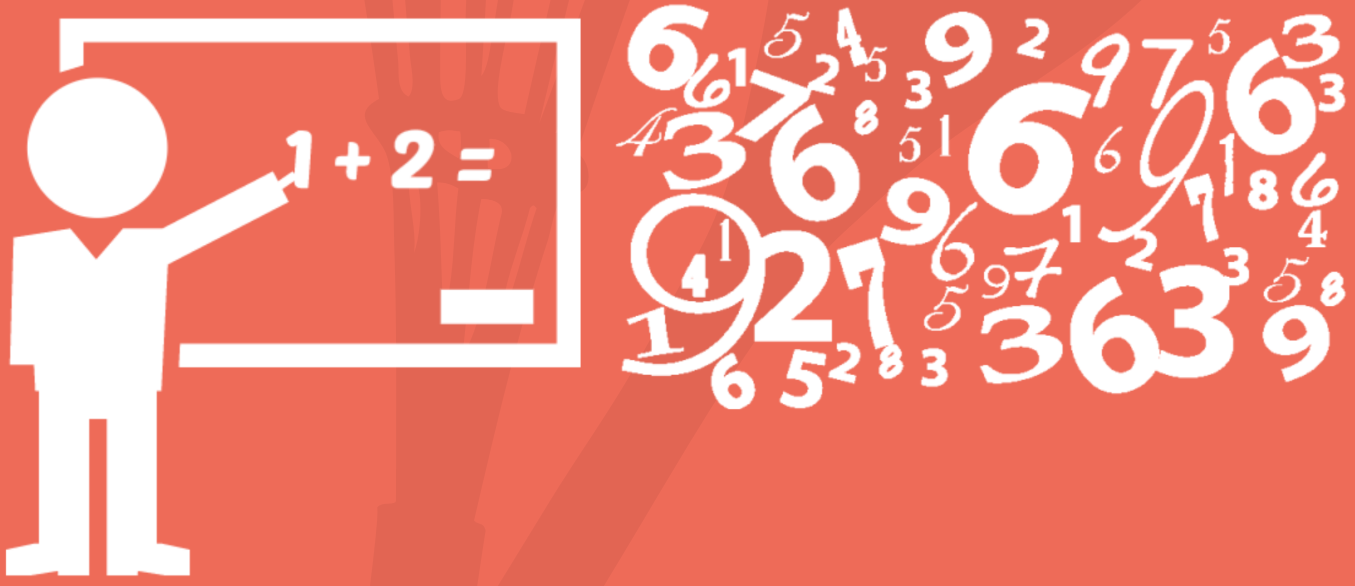
IGCSE Mathematics Revision Guide

New for 2015



Mathematics IGCSE notes

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{also use the intranet revision course of question papers and answers by topic }

1. <u>Decimals and standard form</u>

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(a) multiplying and dividing

(i) $\boxed{2.5\times 1.36}$ Move the decimal points to the right until each is a whole number, noting the total number of moves, perform the multiplication, then move the decimal point back by the previous total:
$\rightarrow 25\times 136 \;=\; 3400$, so the answer is $\underline{3.4}$

{Note in the previous example, that transferring a factor of 2, or even better, 4, from the 136 to the 25 makes it easier:
$25\times 136 = 25\times(4\times 34) \;=\; (25\times 4)\times 34 \;=\; 100\times 34 \;=\; 3400$ *}*

(ii) $\boxed{0.00175\div 0.042}$ Move both decimal points together to the right until the <u>divisor</u> is a whole number, perform the calculation, and that is the answer.
$\rightarrow 1.75\div 42$, but simplify the calculation by cancelling down any factors first. In this case, both numbers share a 7, so divide this out: $\rightarrow 0.25\div 6$, and

$$\begin{array}{r} 0.041\dot{6} \\ 6\overline{)\;0.25\;} \end{array}$$

, so the answer is $\underline{0.041\dot{6}}$

(iii) $\boxed{\text{decimal places}}$
To round a number to n d.p., count n digits to the right of the decimal point. If the digit following the n^{th} is ≥ 5, then the n^{th} digit is raised by 1.
e.g. round 3.012678 to 3 d.p. $3.012678 \rightarrow 3.012|678$ so $\underline{3.013}$ to 3 d.p.

(iv) $\boxed{\text{significant figures}}$
To round a number to n s.f., count digits from the left starting with the first non-zero digit, then proceed as for decimal places.
e.g. round 3109.85 to 3 s.f., $3109.85 \rightarrow 310|9.85$ so $\underline{3110}$ to 3 s.f.
e.g. round 0.0030162 to 3 s.f., $0.0030162 \rightarrow 0.00301|62$, so $\underline{0.00302}$ to 3 s.f.

(b) standard form

(iii) $\boxed{\text{Convert}}$ the following to standard form: (a) 25 000 (b) 0.0000123
Move the decimal point until you have a number x where $1\leq x<10$, and the number of places you moved the point will indicate the numerical value of the power of 10. So $25000 = \underline{2.5\times 10^4}$, and $0.0000123 = \underline{1.23\times 10^{-5}}$

(iv) $\boxed{\text{multiplying}}$ in standard form: $(4.4\times 10^5)\times\left(3.5\times 10^6\right)$ As all the elements are multiplied, rearrange them thus:
$=(4.4\times 3.5)\times\left(10^5\times 10^6\right) \;=\; 15.4\times 10^{11} \;=\; \underline{1.54\times 10^{12}}$

(v) dividing in standard form: $\dfrac{3.2\times10^{12}}{2.5\times10^{3}}$ Again, rearrange the calculation to $(3.2\div2.5)\times(10^{12}\div10^{3}) \;=\; \underline{1.28\times10^{9}}$

(vi) adding/subtracting in standard form: $(2.5\times10^{6})+(3.75\times10^{7})$ The hardest of the calculations. Convert both numbers into the same denomination, i.e. in this case 10^6 or 10^7, then add.

$=(0.25\times10^{7}) + (3.75\times10^{7}) \;=\; \underline{4\times10^{7}}$

<u>Questions</u>

(a) 2.54×1.5

(b) $2.55\div0.015$

(c) Convert into standard form and multiply: $25\,000\,000\times0.000\,000\,000\,24$

(d) $(2.6\times10^{3})\div(2\times10^{-2})$

(e) $(1.55\times10^{-3})-\left(2.5\times10^{-4}\right)$

<u>Answers</u>

(a) $\rightarrow 254\times15 \;=\; 3810$, so $2.54\times1.5 = \underline{3.81}$

(b) $2.55\div0.015 = 2550\div15$. Notice a factor of 5, so let's cancel it first:
$=510\div3 \;=\; \underline{170}$

(c) $=(2.5\times10^{7})\times(2.4\times10^{-10}) \;=\; \underline{6\times10^{-3}}$

(d) $=(2.6\div2)\times(10^{3}\div10^{-2}) \;=\; \underline{1.3\times10^{5}}$

(e) $=(1.55\times10^{-3})-(0.25\times10^{-3}) \;=\; \underline{1.3\times10^{-3}}$

2. Accuracy and Error

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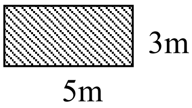
To see how error can accumulate when using rounded values in a calculation, take the worst case each way: e.g. this rectangular space is measured as 5m by 3m, each measurement being to the nearest metre. What is the area of the rectangle?



To find how small the area could be, consider the lower bounds of the two measurements: the length could be as low as 4.5m and the width as low as 2.5m. So the smallest possible area is $4.5 \times 2.5 = 11.25\text{ m}^2$. Now, the length could be anything up to 5.5m but not including the value 5.5m itself (which would be rounded up to 6m) So the best way to deal with this is to use the (unattainable) upper bounds and get a ceiling for the area as $5.5 \times 3.5 = 19.25\text{ m}^2$, which the area could get infinitely close to, but not equal to. Then these two facts can be expressed as $11.25\text{m}^2 \le area < 19.25\text{m}^2$.

Questions

(a) A gold block in the shape of as cuboid measures 2.5cm by 5.0cm by 20.0cm, each to the nearest 0.1cm. What is the volume of the block?

(b) A runner runs 100m, measured to the nearest metre, in 12s, measured to the nearest second. What is the speed of the runner?

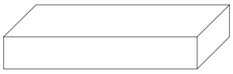
(c) $a = 3.0, \quad b = 2.5$, both measured to 2 s.f. What are the possible value of $a - b$?

Answers

(a) lower bound volume $= 2.45 \times 4.95 \times 19.95 = 241.943625\text{ cm}^3$.

upper bound volume $= 2.55 \times 5.05 \times 20.05 = 258.193875\text{ cm}^3$.

So $\underline{241.943625\text{cm}^3 \le \text{volume} < 258.193875\text{cm}^3}$



(b) Since speed $= \dfrac{\text{distance}}{\text{time}}$, for the lower bound we need to take the smallest value of distance with the biggest value of time, and vice-versa for the upper bound.

So $\dfrac{99.5}{12.5} < \text{speed} < \dfrac{100.5}{11.5}$, i.e. $\underline{7.96\text{ms}^{-1} \le \text{speed} < 8.739\ldots\text{ms}^{-1}}$

(c) for the smallest value of $a - b$, we need to take the smallest value of a together with the biggest value of b, etc.

So $2.95 - 2.55 < a - b < 3.05 - 2.45$, i.e. $\underline{0.4 < a - b < 0.6}$

3. **Powers and roots**

top

1) $x^a \times x^b = x^{a+b}$
2) $x^a \div x^b = x^{a-b}$
3) $(x^a)^b = x^{ab}$
4) $x^{-a} = \dfrac{1}{x^a}$
5) $x^0 = 1$

(a) whole number powers

Note that the base numbers (x's) have to be the same;
$2^5 \times 3^2$ cannot be simplified any further.

1) e.g. $x^3 \times x^2 = x^5$, $\quad 2^3 \times 2^7 = 2^{10}$

If in doubt, write the powers out in full: $a^3 \times a^2$ *means* $(a \times a \times a) \times (a \times a)$ *which is* a^5

2) $x^6 \div x^2 = x^4$, $\quad 5^8 \div 5^2 = 5^6$

Again, if in doubt, spell it out:

$a^6 \div a^2$ *means* $\dfrac{a \times a \times a \times a \times a \times a}{a \times a}$ *which cancels down to*

$$\frac{a \times a \times a \times a \times \cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a}} = a^4$$

3) $(x^3)^2 = x^6$, $\quad (3^2)^4 = 3^8$

To check this, $(x^3)^2$ *means* $(x^3) \times (x^3)$ *which is* x^6

4) $x^{-3} = \dfrac{1}{x^3}$, $\qquad 10^{-3} = \dfrac{1}{10^3} = \dfrac{1}{1000} = 0.001$

5) $10^0 = 1$.

Questions

Simplify the following as far as possible:

(a) $x^5 \times x^3$

(b) $a^3 \div a^5 \times a^6$

(c) $\dfrac{3^4 \times 3^7}{3^5 \times 3}$

(d) $\dfrac{2^5 \times 4^{10}}{8^6 \div 4^3}$

Answers

(a) $x^5 \times x^3 = x^{5+3} = x^8$

(b) $a^3 \div a^5 \times a^6 = a^{3-5+6} = a^4$

(c) $\dfrac{3^4 \times 3^7}{3^5 \times 3} = \dfrac{3^{4+7}}{3^{5+1}} = \dfrac{3^{11}}{3^6} = 3^{11-6} = 3^5$

(d) $\dfrac{2^5 \times 4^{10}}{8^6 \div 4^3} = \dfrac{2^5 \times (2^2)^{10}}{(2^3)^6 \div (2^2)^3} = \dfrac{2^5 \times 2^{20}}{2^{18} \div 2^6} = \dfrac{2^{25}}{2^{12}} = 2^{13}$

(b) fractional powers

6) $x^{\frac{1}{n}} = \sqrt[n]{x}$

7) $x^{\frac{p}{q}} = \sqrt[q]{x^p} = (\sqrt[q]{x})^p$

6) e.g. $x^{\frac{1}{3}} = \sqrt[3]{x}$, $9^{\frac{1}{2}} = \sqrt{9} = 3$

7) $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = (3)^2 = 9$

Note: if you can find the qth root of x easily then it's better to use the $(\sqrt[q]{x})^p$ version.

Q. Simplify the following as far as possible:

(a) $16^{\frac{1}{2}}$

(b) $64^{\frac{1}{3}}$

(c) $4^{\frac{3}{2}}$

(d) $81^{\frac{3}{4}}$

(e) $(x^6)^{\frac{2}{3}}$

Answers.

(a) $16^{\frac{1}{2}} = \sqrt{16} = 4$

(b) $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

(c) $4^{\frac{3}{2}} = (\sqrt{4})^3 = (2)^3 = 8$

(d) $81^{\frac{3}{4}} = (\sqrt[4]{81})^3 = (3)^3 = 27$

(e) *easier to use power law (3) above:* $(x^6)^{\frac{2}{3}} = x^{6\times\frac{2}{3}} = x^4$

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4. Ratio and Proportion

(a) Using ratios

This is really a special case of proportion. If quantities are linearly related, either directly or inversely, (like number of workers and time taken to do a job), calculate by multiplying by a ratio:

e.g. If 8 workers can together do a job in 6 days, how long would the same job take with 12 workers?
ans: it will take less time, so we multiply by the ratio $\frac{8}{12}$. So it takes $6\times\frac{8}{12} = 4$ days.

e.g. If a workforce of 20 can produce 12 cars in 15 days, how many workers should be used if 15 cars are needed in 10 days?
ans: no. of workers = $20\times\frac{15}{12}\times\frac{15}{10} = 37\frac{1}{2}$, ie 38.

(b) Proportion

Where quantities are related not necessarily linearly.

(i) Direct proportion

This is when an **increase** in one quantity causes an **increase** in the other.

eg $y \propto x^2$, and you are given that y is 7.2 when x is 6.

Rewrite as $y = kx^2$, and substitute the given values to find k:

$7.2 = k\times 6^2$, so

$k = 0.2$. The relationship can now be written as

$y = 0.2x^2$, and any problems solved.

(ii) Inverse proportion

This is when an **increase** in one quantity causes an **decrease** in the other.
e.g. If y is inversely proportional to the cube of x,

then $y \propto \dfrac{1}{x^3}$

Rewrite as $y = \dfrac{k}{x^3}$, and proceed as usual.

(iii) Multiplier method

For the cunning, it is possible (but harder) to solve a problem without calculating k.
e.g. Radiation varies inversely as the square of the distance away from the source. In suitable units, the radiation at 10m away from the source is 75. What is the radiation at 50m away?
ans: as distance increases by a factor of 5, radiation must decrease by a factor of 5^2, so the radiation is $75 \div 25 = 3$.

Questions.

(a) Water needs to be removed from an underground chamber before work can commence. When the water was at a depth of 3m, five suction pipes were used and emptied the chamber in 4 hours. If the water is now at a depth of 5m (same cross-section), and you want to empty the chamber in 10 hours time, how many pipes need to be used?

(b) *y* is proportional to x^2 and when *x* is 5 *y* is 6. Find
(i) *y* when *x* is 25 (ii) *x* when *y* is 8.64

(c) The time *t* seconds taken for an object to travel a certain distance from rest is inversely proportional to the square root of the acceleration *a*. When *a* is 4m/s^2, t is 2s.
What is the value of *a* if the time taken is 5 seconds?

Answers

(a) No. of pipes = $5\times\dfrac{5}{3}\times\dfrac{4}{10} = 3\tfrac{1}{3}$, so it would be necessary to use 4 pipes to be sure of emptying within 10 hours.

(b) $y \propto x^2$ and we know when *x* is 5, *y* is 6, so
$y = kx^2$

$6 = k\times 5^2$, so $k = \frac{6}{25}$, and we can write the relationship as
$y = \frac{6}{25}x^2$.

(i) When *x* is 25, $y = \frac{6}{25}\times 25^2 = 150$.

(ii) When *y* is 8.64, $8.64 = \frac{6}{25}x^2$, so

$x^2 = \dfrac{25\times 8.64}{6}$ no, don’t reach for the calculator yet!

$x^2 = 25\times 1.44$, so $x = 5\times 1.2 = 6$.

(c) $t \propto \dfrac{1}{\sqrt{a}}$,

So $t = \dfrac{k}{\sqrt{a}}$. Substituting given values:

$2 = \dfrac{k}{\sqrt{4}}$, so $k = 4$, ie

$t = \dfrac{4}{\sqrt{a}}$.

When t = 5, $5 = \dfrac{4}{\sqrt{a}}$, so $\sqrt{a} = \dfrac{4}{5}$, and $a = \dfrac{16}{25}$ or 0.64 m/s^2.

5. Fractions and ratios

top

(a) Fractions

(i) Adding/subtracting: e.g. $3\frac{1}{6}-1\frac{2}{3}$. Convert to vulgar form first: $\frac{19}{6}-\frac{5}{3}$, then find the lowest common denominator, in this case 6. Then

$$\frac{19}{6}-\frac{5}{3}=\frac{19-2\times5}{6} = \frac{9}{6} = 1\frac{1}{2}.$$

(ii) Multiplying/dividing: e.g. $5\frac{1}{3}\times\frac{7}{8}$. Convert to vulgar form: $\frac{16}{3}\times\frac{7}{8}$, and then always cancel any factor in the numerator with a factor in the denominator if possible, before multiplying together:

$$\frac{16}{3}\times\frac{7}{8} = \frac{{}^{2}\cancel{16}}{3}\times\frac{7}{{}^{1}\cancel{8}} = \frac{2\times7}{3\times1} = \frac{14}{3}.$$

To divide, turn the ÷ into a × and invert the <u>second</u> fraction.

(iii) Converting to and from decimals: e.g. what is $\frac{3}{40}$ as a decimal?

$\begin{array}{r}0.075\\ 40\overline{)3.000}\end{array}$ so $\frac{3}{40}$ is <u>0.075</u>.

But what is 0.075 as a fraction? 0.075 means $\frac{75}{1000}$, then cancel down to $\frac{3}{40}$.

(b) Ratios

(iv) To divide a quantity into 3 parts in the ratio 3: 4:5, call the divisions 3 parts, 4 parts and 5 parts. There are 12 parts altogether, so find 1 part, and hence the 3 portions.

(v) To find the ratio of several quantities, express in the same units then cancel or multiply up until in lowest terms e.g. what is the ratio of 3.0m to 2.25m to 75cm?
Perhaps metres is the best unit to use here, so the ratio is 3 :2.25: 0.75. Multiplying up by 4 (or 100 if you really insist) will render all numbers integer. So the ratio is 12 : 9 : 3, and we can now cancel down to <u>4:3:1</u>

Questions

(a) $(2\frac{3}{4})^2 \times 1\frac{5}{11}$

(b) $(1\frac{1}{3} - \frac{3}{5}) \div 2\frac{1}{5}$

Answers

(a) $= (\frac{11}{4})^2 \times \frac{16}{11} \quad = \quad \frac{121}{16} \times \frac{16}{11} \quad = \quad \frac{{}^{11}\cancel{121}}{{}^{1}\cancel{16}} \times \frac{{}^{1}\cancel{16}}{{}^{1}\cancel{11}} \quad = \quad 11$

(b) $= (\frac{4}{3} - \frac{3}{5}) \div \frac{11}{5} \quad = \quad \frac{5 \times 4 - 3 \times 3}{15} \times \frac{5}{11} \quad = \quad \frac{11}{15} \times \frac{5}{11} \quad = \quad \frac{1}{3}$

(c) $= \frac{875}{10\,000} \quad = \quad \frac{35}{400} \quad = \quad \frac{7}{80}$

(d) 1:2:5 means 8 parts altogether. Each part is $£5000 \div 8 \quad = \quad £625$, so the £5000 splits into £625, £1250, and £3125.

6. **Percentages**

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(i) What is 75g as a percentage of 6kg? Express as a fraction, then multiply by 100 to covert to a percentage. As a fraction, it is $\frac{75}{6000}$, which is

$$\frac{75}{6000}\times 100\% \;=\; \frac{75}{60}\% \;=\; 1\frac{1}{4}\%.$$

(ii) Find 23% of 3.2kg. This is

$$\frac{23}{100}\times 3.2\text{kg} \;=\; \frac{23}{100}\times 3200\text{g} \;=\; 23\times 32\text{g} \;=\; 736\text{g} \;\text{(or 0.736kg.)}$$

(iii) Increase £20 by 12%. The original amount is always regarded as 100%, and this problem wants to find 112%. The simplest method is to first find 1%, then 112%, by dividing by 100 then multiplying by 112. This can be accomplished in one go, however, by multiplying by $\frac{112}{100}$, i.e. 1.12.

So the answer is $£20\times 1.12 \;=\; £22.40$.

(iv) Decrease £20 by 12%. This means we are trying to find 88% of the original, so the answer is $£20\times 0.88 \;=\; £17.60$.

(v) Reverse problems: An investment is worth £6000 after increasing by 20% in a year: how much was it worth last year? If you are going to make a mistake, this is where. The 20% refers to 20% of the original amount which we don't know, not 20% of £6000. A safe way of handling these "reverse" problems is to call the unknown original amount £*x*. The information says that

$$£x\times 1.2 = £6000 \;\text{ so }\; x = \frac{6000}{1.2} = 5000.$$

(vi) Anything weird, and use the simple unitary method, i.e. find what is 1%. e.g. A coke can advertises 15% extra free, and contains 368ml. How much extra coke was there?

This can contains 115% of the original, so 1% is $368 \div 115 \;=\; 3.2$ ml.
So the extra amount, 15%, is $15\times 3.2 \;=\; 48$ ml.

Questions

(a) One part of a company produces £350 000 profit, while the whole company makes £5.6 million. What percentage of the whole company's profits does this part produce?

(b) How much VAT at 17½% is added to a basic price of £25?

(c) An investment earns 8% interest every year. My account has £27000 this year. How much is contained in my account (i) next year (ii) in ten years' time (iii) last year?

(d) Inflation runs at 4% per year in Toyland. Big Ears can buy 24 toadstools for £1 this year. How many will he be able to buy for £1 in 5 years' time?

Answers

(a) $\dfrac{350\,000}{5\,600\,000} \times 100\% \;=\; 6\dfrac{1}{4}\%$

(b) 17½% of £25 is $\dfrac{17\frac{1}{2}}{100} \times 25 \;=\; \dfrac{175}{1000} \times 25 \;=\; £4.38$

(c) (i) $£27000 \times 1.08 \;=\; £29160$

(ii) $£27000 \times 1.08^{10} \;=\; £58290.97$

(iii) $£x \times 1.08 = £27000 \qquad x = \dfrac{27000}{1.08}$, so it was worth £25000.

(d) Inflation at 4% per year means that if you pay £100 for some goods this year, the same goods will cost you £104 in next years' money. So 24 toadstools will cost $£1 \times 1.04^{5} \;=\; £1.2166529...$ in 5 years' time, and so £1 will buy him $24 \times \dfrac{1}{1.2166529...}$, i.e. 19.7… or 19 whole toadstools!

7. Rational and irrational numbers

top

A rational number is one which can be expressed as $\frac{a}{b}$ where *a* and *b* are integers. An irrational number is one which can't. Fractions, integers, and recurring decimals are rational. Examples of rationals: $\frac{2}{3}$, 1, 0.25, $\sqrt[3]{8}$.

Examples of irrationals: π, $\sqrt{2}$, 0.1234....(not recurring).

(i) Converting rationals to the form $\frac{a}{b}$ (to confirm they really are rational)

A terminating decimal: $0.125 \;=\; \frac{125}{1000} \;=\; \frac{1}{8}$

A recurring decimal: $0.\dot{1}2\dot{3}$. Call the number *x*, so $x = 0.123123123......$
Multiply by a suitable power of 10 so the recurring decimal appears exactly again: $1000x = 123.123123.....$ $\quad = 123 + 0.123123....$

so $1000x = 123 + x$, then $999x = 123$ and $x = \frac{123}{999} = \frac{41}{333}$.

(ii) rationalising a denominator:

$3\frac{\sqrt{2}}{\sqrt{3}}$ has a $\sqrt{3}$ in the denominator, so multiply top and bottom by $\sqrt{3}$ (which does not change the value of the expression, only the shape):

$$3\frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \;=\; 3\frac{\sqrt{6}}{3} \;=\; \sqrt{6}\,.$$

(iii)

$$\boxed{\begin{array}{l} \sqrt{a}\sqrt{b} = \sqrt{ab} \\ \dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}} \end{array}}$$

and the same with cube roots, etc.
To simplify expressions using these:

$$\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} \;=\; 10\sqrt{2}$$

$$\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} \;=\; 3$$

(iv) Finding irrational numbers in a given area:
e.g. find an irrational number between 5 and 6. Note that most square roots are irrational (except for $\sqrt{16}$, $\sqrt{\frac{4}{9}}$, etc) are irrational, so as $5 = \sqrt{25}$ and $6 = \sqrt{36}$, pick a root in between, e.g. $\sqrt{28}$. (Or say $\pi + 2$ for example).

Questions

(a) Convert into the form $\frac{a}{b}$:   (i) 0.375   (ii) $0.\dot{3}\dot{6}$

(b) Simplify (i) $\frac{6}{\sqrt{2}}$ (ii) $\frac{\sqrt{50}}{\sqrt{2}}$ (iii) $\sqrt{72}$ (iv) $\sqrt[3]{250}$

(c) Find an irrational number between 1 and 1.1

Answers

(a) (i) $0.375 = \frac{375}{1000} = \frac{3}{8}$ (ii) $x = 0.36363636......$ so $100x = 36.363636.....$

$= 36 + 0.363636 = 36 + x$. Therefore $99x = 36$, so $x = \frac{36}{99} = \frac{4}{11}$

(b) (i) $= \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

(ii) $= \sqrt{\frac{50}{2}} = \sqrt{25} = 5$

(iii) $\sqrt{72} = \sqrt{36 \times 2} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$

(iv) $= \sqrt[3]{125 \times 2} = \sqrt[3]{125} \times \sqrt[3]{2} = 5\sqrt[3]{2}$

(c) e.g. $\sqrt{2} - 0.3$, $\frac{\sqrt{101}}{10}$ etc

8. Algebra:

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(a) Simplifying

Multiply out brackets and gather up like terms: e.g.
$3x(x+2y)-2y(3x-y) \;=\; 3x^2+6xy-6xy+2y^2 \;=\; 3x^2+2y^2$

(b) Factorising

(i) extracting the highest common factor: $6x^2-3xy \;=\; 3x(2x-y)$
(multiply out the answer to check)

(ii) quadratics:

(a) $\boxed{x^2-2x}$ (no number term): $=x(x-2)$

(b) $\boxed{x^2-16}$ (no x term): if it is difference of two squares as in this case:
$=(x-4)(x+4)$

(c) $\boxed{x^2-3x-4}$ (a full quadratic): start with $(x\quad)(x\quad)$ form, then look for two numbers which multiply to give -4 and add to give -3. These are -4 and $+1$. So $(x-4)(x+1)$ is the answer. *(multiply out the answer to check!)*

(d) $\boxed{2x^2+9x+4}$ (full quadratic with more than one x^2): multiply the 2 by the 4 to get 8, and repeat the previous process i.e. look for two numbers which multiply to 8 and add up to 9. These are +8 and+1. Now split the middle term accordingly and group into 2 pairs:
$2x^2+9x+4=2x^2+8x+x+4=(2x^2+8x)+(x+4)$ Then factorise each group, $=2x(x+4)+(x+4)$, and notice the bracket factor which you now extract: $=(x+4)(2x+1)$.

(iii) grouping: unusual, but reminiscent of part of (d) above, expressions like $ab+ac-b^2-bc$ may be able to be factorised even though there are apparently no common factors. $=(ab+ac)-(b^2+bc) \quad =a(b+c)-b(b+c)$, and there just happens to be a big factor: $=(b+c)(a-b)$

Questions

(a) Simplify $a(b-c)+b(c-a)+c(a-b)$

(b) Factorise (i) $4p^2q-6pq$ (ii) $2x^2+6x$ (iii) $4x^2-1$
(iv) $x^2+10x+21$ (v) $3x^2+11x+6$ (vi) $2ab-6ac+b-3c$

Answers

(a) $=ab-ac+bc-ab+ac-bc \;=\; 0$

(b) (i) $=2pq(2p-3)$ (ii) $=2x(x+3)$ (iii) $=(2x-1)(2x+1)$
(iv) $=(x+7)(x+3)$ (v) 3×6 gives 18, so 9 and 2 are the required numbers:
$3x^2+9x+2x+6 \;=(3x^2+9x)+(2x+6) \;=3x(x+3)+2(x+3)$ and finally
$=(3x+2)(x+3)$.
(vi) $=2a(b-3c)+(b-3c) \;=(2a+1)(b-3c)$

9. Equations: linear, quadratic, simultaneous

top

(a) Linear

Perform the same operation on both sides to isolate x:

$$\frac{2x}{3}+x=\frac{1}{2} \qquad [\times 6]$$
$$4x+6x=3$$
$$10x=3 \qquad [\div 10]$$
$$x=\frac{3}{10}.$$

(b) Quadratic

(i) rearrange the equation if necessary to get 0 on the right.

If it can be factorised, do so (see 8. Algebra). Then:
$(2x-1)(x+3)=0$ means one of the brackets must be 0, so
$2x-1=0$ or $x+3=0$, which can be solved to give
$x=\frac{1}{2},\ -3$

If not, use $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ and round the answers suitably.

(c) Simultaneous

2 linear equations:

(i) elimination

Multiply both equations until either the x's or the y's are the same then add/subtract so that they disappear.

$2x-y=7$
$3x+2y=5$ multiply equation 1 by 2, then add:

$$\begin{array}{l} 4x-2y=14 \\ \underline{3x+2y=5} \\ 7x=19 \end{array}$$

solve and substitute back in to equation 1 to find y.

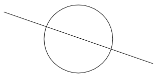
(ii) substitution
isolate x or y from one equation and substitute its value into the other:
$2x-3y=5$
$y=5x-2$
Use the expression for y in equation 2 and substitute it into equation 1:
$2x-3(5x-2)=5$, and proceed.

one linear, one quadratic:

$x^2 + y^2 = 25$

$x + y = 0.8$

Rearrange the linear equation and substitute into the quadratic:
$y = 0.8 - x$, so $x^2 + (0.8 - x)^2 = 25$. Multiply out, and solve the quadratic in *x*.
Note that each *x* answer will then produce a *y* answer, and this gives two pairs, as it should because the equations represent the intersection of :



Questions

(a) Solve $\frac{x}{3} - \frac{1-x}{2} = 1$

(b) Solve $x^2 + 2x = 15$

(c) Solve $2x^2 + x - 6 = 0$

(d) Solve $x - \frac{1}{x} = 2$

(e) Solve the simultaneous equations $\begin{aligned} x + 2y &= 5 \\ x^2 - y^2 &= -3 \end{aligned}$

Answers

(a) $\frac{x}{3} - \frac{1-x}{2} = 1$ $\quad [\times 6]$

$2x - 3(1-x) = 6$ $\quad \rightarrow 2x - 3 + 3x = 6$

$5x = 9$, so $x = \frac{9}{5}$.

(b) $x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

$x = -5,\ \ 3$.

(c) $2 \times -6 = -12$, so look for two numbers which multiply to -12 and add to 1. These are 4, -3.

So $2x^2 + 4x - 3x - 6 = 0$

$(2x^2 + 4x) - (3x + 6) = 0$

$2x(x+2) - 3(x+2) = 0$

$(2x-3)(x+2) = 0$, which gives $x = -2, \frac{3}{2}$.

(d) $x-\dfrac{1}{x}=2 \quad [\times x]$

$x^2-1=2x$

$x^2-2x-1=0$

$x=\dfrac{2\pm\sqrt{(-2)^2-4\times1\times-1}}{2\times1} \qquad =\dfrac{2\pm\sqrt{8}}{2}$ {Note that $\sqrt{8}=\sqrt{4\times2}=2\sqrt{2}$ and so 2 can be cancelled}: $=1\pm\sqrt{2}$, so $x=$ <u>-0.41, 2.41</u> to 2 d. p.

(e) $\begin{aligned} x+2y&=5 \\ x^2-y^2&=-3 \end{aligned}$ rearrange equation1 : $x=5-2y$, and substitute:

$\therefore (5-2y)^2-y^2=-3$

$\therefore 25-20y+4y^2-y^2=-3$

$\therefore 3y^2-20y+28=0$

$\therefore (3y-14)(y-2)=0$

$\therefore y=\dfrac{14}{3},\ 2$. These lead to $x=-\dfrac{13}{3},\ 1$, so the two answers are

$(x,y)=(-\dfrac{13}{3},\dfrac{14}{3}),\ \ (1,2)$.

10. Rearranging formulae

top

(i) with a variable which only appears once, treat like an equation and isolate the variable: e.g. make x the subject of $\dfrac{ax+b}{c}=d$: $[\times c]$ gives $ax+b=cd$, $[-b]$ gives $ax=cd-b$, and finally $[\div a]$ gives $x=\dfrac{cd-b}{a}$.

(ii) with a variable which appears more than once, gather together and factorise: e.g. $ax=bx+c$ $[-bx]$ gives $ax-bx=c$, factorising gives $(a-b)x=c$, then $[\div(a-b)]$ gives $x=\dfrac{c}{a-b}$.

<u>Questions</u>

(a) The Centigrade and Fahrenheit scales are related linearly by $C=\frac{9}{5}(F-32)$. Rearrange it to make F the subject.

(b) Make x the subject of $\dfrac{x-a}{x}=b$

<u>Answers</u>

(a) $C=\frac{9}{5}(F-32)$ $\quad[\times 5]$

$\therefore 5C=9(F-32)$ $\quad[\div 9]$

$\therefore \dfrac{5C}{9}=F-32$ $\quad[+32]$

$\therefore F=\frac{5}{9}C+32$

(there are different ways to approach this, but all (correct) answers will be equivalent even though they may look different)

(b) $\dfrac{x-a}{x}=b$ $\quad[\times x]$

$\therefore x-a=bx$ $\quad[-bx,+a]$

$\therefore x-bx=a$ $\quad[\mathit{factorise}]$

$\therefore x(1-b)=a$ $\quad[\div(1-b)]$

$\therefore x=\dfrac{a}{1-b}$.

{ Note that $x=\dfrac{-a}{b-1}$ would also be correct, as top and bottom are multiplied by –1}

11. Inequalities

top

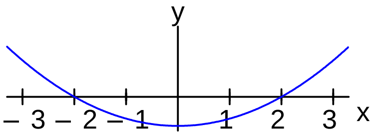
(a) linear

treat exactly like an equation, except if you multiply/divide by a negative number, the inequality sign must be reversed. Avoid it!

(b) quadratic

e.g. $x^2 < 4$ First treat like an equation and factorise if possible (formula otherwise): $x^2 - 4 < 0$, then $(x-2)(x+2) < 0$. This gives "critical values" of –2 and +2. Draw a number line, and a sketch of the function (in this case a "happy" parabola) which reveals the region in which $x^2 - 4 < 0$:

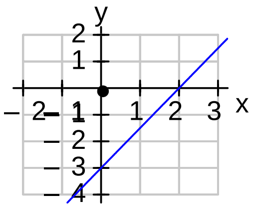
$\underline{-2 < x < 2}$.



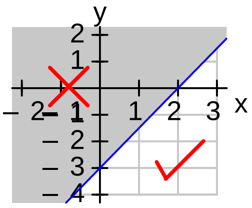
Had the question been $x^2 > 4$, the answer would be $\underline{x < 2}$ **or** $\underline{x > 2}$.

(c) 2 variable linear inequalities

e.g. $3x - 2y \geq 6$. Plot the boundary line $3x - 2y = 6$, then take a trial point (e.g. the origin) to determine which side of the line to accept.



The origin's coordinates make $3 \times 0 - 2 \times 0$ which is not ≥ 6, so that side is rejected:

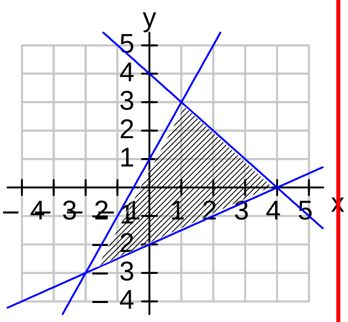


Questions

(a) Solve $2(1-x)<6$

(b) Solve $12-x \le x^2$

(c) Find the 3 inequalities which identify this region:



Answers

$$2(1-x)<6 \qquad [\div 2]$$

(a) $2(1-x)<6$ [$\div 2$]

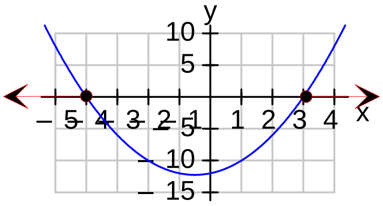
$\therefore 1-x<3$ [$+x,\ -3$]

$\underline{x>-2}$

(b) $12-x \le x^2$ [rearrange]

$x^2+x-12 \ge 0$

$\therefore (x+4)(x-3) \ge 0$, giving critical values of –4 and +3.



so $\underline{x \le -4}$ or $\underline{x \ge 3}$

(c) The three line equations are $y=2x+1,\ y=\frac{1}{2}x-2,\quad x+y=4$.

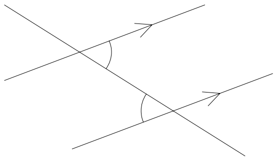
By considering a point (e.g. origin) in the shaded region, the inequalities are

$y<2x+1$, $y>\frac{1}{2}x-2$, and $x+y<4$.

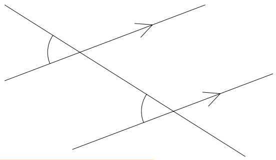
12. Parallel lines, bearings, polygons

top

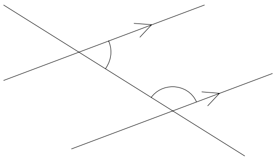
(a) Parallel lines



alternate angles equal



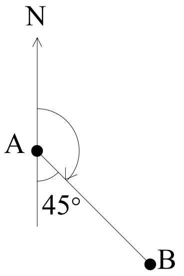
corresponding angles equal



allied or interior add up to 180°

(b) bearings

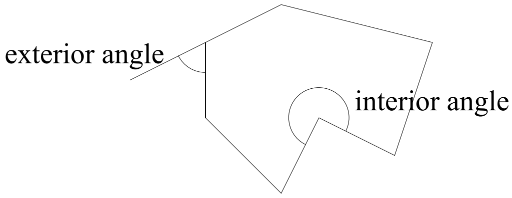
Bearings are measured clockwise from North:
bearing of B from A is 135°



(c) polygons

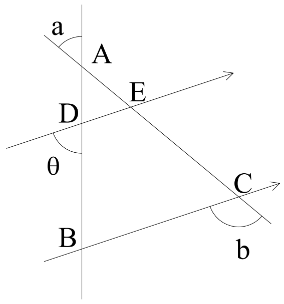
for a polygon with *n* sides,
sum of interior angles = $(n-2)180^{\circ}$

sum of exteriors = 360°



Questions

(a) In the diagram opposite, find the value of θ in terms of a and b.

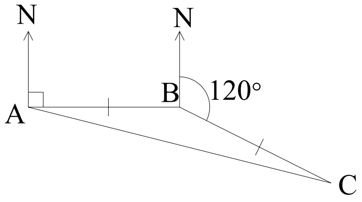


(b) The bearing of B from A is 090º, and the bearing of C from B is 120º. Given also that AB = BC, find the bearing of C from A.

(c) A pentagon has exactly one line of symmetry, and angles all of which are either 100º or 120º. Make a sketch of the pentagon, marking in the angles.

Answers

(a) $D\hat{A}E = a$ (opposite), $D\hat{E}C = b$ (corresponding), so $A\hat{E}D = 180 - b$ (angles on a straight line). $A\hat{D}E = \theta$ (opposite). We now have the three angles in triangle ADE, so $a + (180 - b) + \theta = 180$. A rearrangement gives $\underline{\theta = b - a}$.

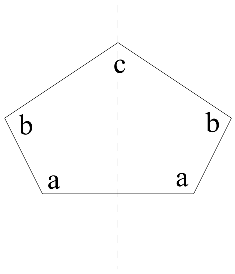


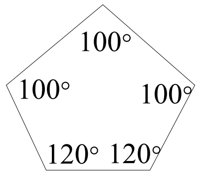
(b) Angle at point B means

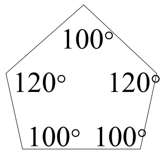
$A\hat{B}C = 360 - 90 - 120 = 150\,^{\circ}$

Triangle ABC is isosceles, so $B\hat{A}C = 15\,^{\circ}$.
The bearing of C from A is therefore <u>105º</u>.

(c) Sum of internal angles is $(n-2)180 = 540\,^{\circ}$ for any pentagon. A line of symmetry means the set up is like this:
The only way of allocating 100º and 120º to *a*, *b*, *c* and make a total of 540º is to have three 100º's and two 120º's. So there are two possible pentagons:





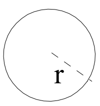


13. Areas and volumes, similarity

top

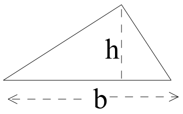
(a) Areas of plane figures

CIRCLE

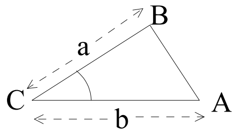


$\boxed{\pi r^2}$

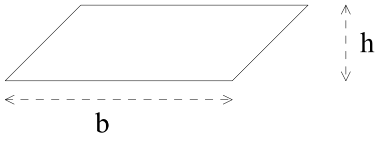
TRIANGLE



$\boxed{\frac{1}{2}bh}$ **or** $\boxed{\frac{1}{2}ab\sin C}$

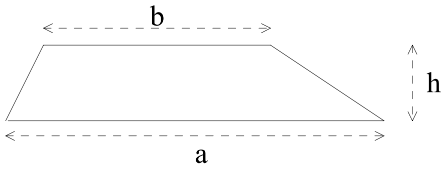


PARALLELOGRAM



$\boxed{bh}$

TRAPEZIUM

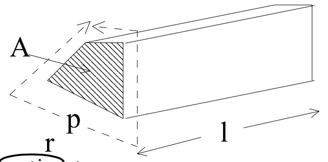


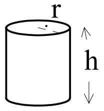
$\boxed{\frac{1}{2}(a+b)h}$

(b) Surface area and volume

Shape	**surface area**	**volume**
Prism	$p \times l$	$\boxed{A \times l}$
Cylinder	$\boxed{2\pi rh}$	$\boxed{\pi r^2 h}$
Cone	$\boxed{\pi rl}$	$\boxed{\frac{1}{3}\pi r^2 h}$
Sphere	$\boxed{4\pi r^2}$	$\boxed{\frac{4}{3}\pi r^3}$
Pyramid		$\boxed{\frac{1}{3} \times \text{base area} \times h}$

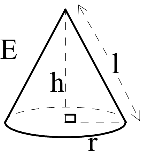
PRISM



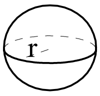


CYLINDER

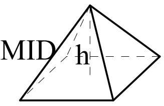
CONE



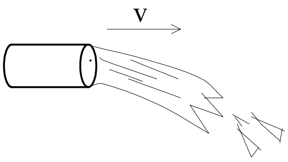
SPHERE



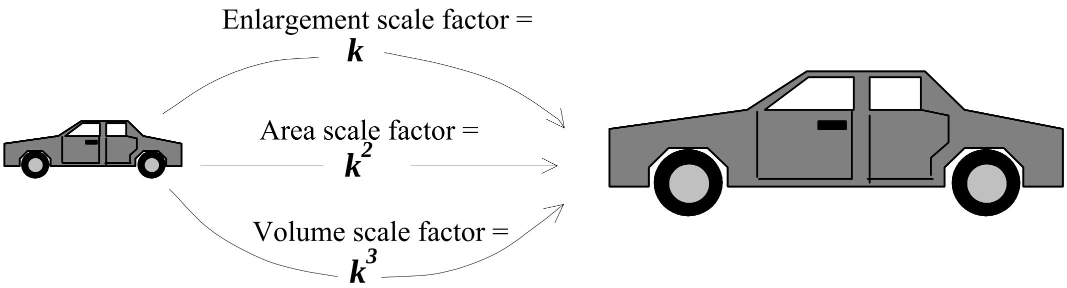
PYRAMID



Pipe flow: number of m^3/s flowing through (or out of) a pipe
$= \boxed{\text{cross-sectional area} \times \text{speed}}$



(b) Similarity

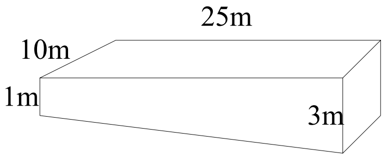


Questions

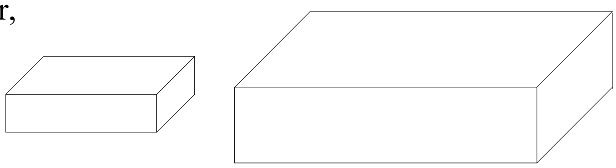
(a) A cylinder has volume 100cm^3, and height 5cm.
What is its diameter?

(b) A cone of base radius 10cm and height 20cm is sliced parallel to the base half way up into two pieces. What is the volume of the base part? (frustum)

(c) The empty swimming pool shown opposite is to be filled with water. The speed of flow of water in the pipe is 2m/s, and the radius of the pipe is 5cm. How long will the pool take to fill?



(d) Two blocks are geometrically similar, and the big blocks weighs 20 times the small block. What is the ratio of surface areas of the two blocks?



Answers

(a) $\pi r^2 5 = 100$

$$\therefore r^2 = \frac{100}{5\pi}, \text{ so } r = \sqrt{\frac{20}{\pi}} = 2.52\,\text{cm}.$$ Whoops! Diameter asked for!

diameter = 5.05cm to 3sf

{Note the pre-corrected value was doubled resulting in 5.05 when itself rounded, not 5.04}

(b) The upper small cone has base radius 5cm and height 10cm. The volume of the base is therefore $\frac{1}{3}\pi 10^2 \times 20 - \frac{1}{3}\pi 5^2 \times 10$ which factorises to $\frac{1}{3}\pi 1750 =$ <u>1830</u>cm^3 to 3sf

(c) Pool is a prism with cross section the side, which is a trapezium.

So volume of pool $= \frac{1}{2}(1+3)25 \times 10 = 500$ m^3.

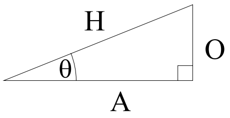
Rate of egress of water is c.s.a. × speed = $\pi 5^2 \times 200 = 5000\pi$ cm^3, which is $5000\pi \div 10^6$ m^3/s. (Units!!) So time taken =

$500 \div (5000\pi \div 10^6) = \frac{10^5}{\pi} = 31831$s, i.e. approx <u>8hrs 51 mins</u>.

(d) assuming same density material, weight is directly proportional to volume. The volume factor is 20, i.e. $k^3 = 20$. $\therefore k = \sqrt[3]{20}$, and so the ratio of surface areas,1 : k^2, is <u>1 : 7.37</u>

14. Trigonometry

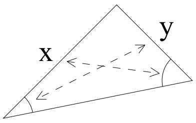
top



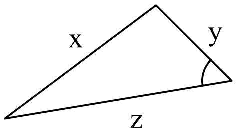
$$\boxed{\begin{aligned}\sin\theta &= \frac{O}{H}\\ \cos\theta &= \frac{A}{H}\\ \tan\theta &= \frac{O}{A}\end{aligned}}$$

Sine rule: $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$

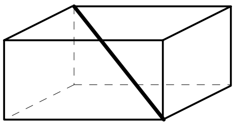
Cosine rule: $\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$



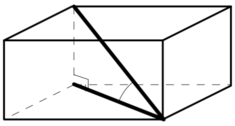
Two opposite pairs: use **sine** rule



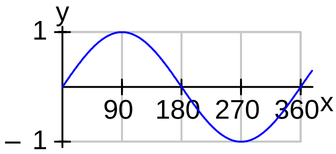
Three sides and one angle: use **cosine** rule



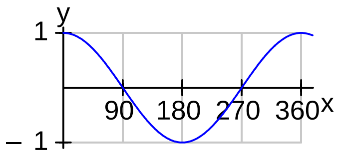
Angle between line and plane is the angle between the line and its projection on the plane: e.g. for the angle between this diagonal and the base, draw the projection, and the angle is shown here:



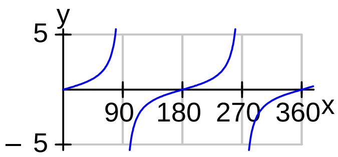
Trigonometric functions for all angles:



sin*x*



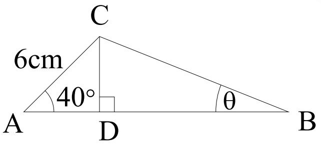
cos*x*



tan*x*

Questions

(a) ADB is a straight line of length 20cm. Find θ.



(b) In triangle ABC, AB = 5cm, BC = 8cm, and $B\hat{C}A = 30°$. Find $A\hat{B}C$

(c) A yacht sails 5 miles at 045° then 6 miles at 090°. How far and at what bearing is it from its original point?

(d) Is an internal diagonal of a cube at 45° elevation from the base?

(e) Find two values of x in the range 0° to 360° for which $\sin x = -0.5$

Answers

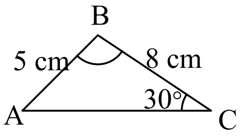
(a) draw a diagram. No, a decent diagram!
θ lies in the triangle on the right, and all the information we have is in the left triangle. To connect with the triangle BCD it would be helpful to calculate CD and AD.

$$\sin 40^\circ = \frac{CD}{6},\ \therefore CD = 6\sin 40^\circ = 3.85....$$

$$\cos 40^\circ = \frac{AD}{6},\ \therefore AD = 6\cos 40^\circ\ = 4.59.....,\ \text{so BD} = 20 - 4.59... = 15.4..$$

Then $\tan\theta = \dfrac{3.85...}{15.4...} = 0.250...$, so $\underline{\theta = 14.1^\circ}$ to 3 s.f.

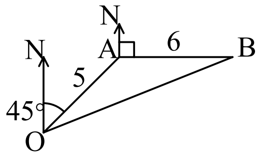
(b) If we target angle A then we have 2 opposite side/angle pairs, so use the sine rule:



$$\frac{\sin A}{8} = \frac{\sin 30^\circ}{5},\ \text{so}\ \hat{A} = 53.1..^\circ,\ \text{and}$$

$$\hat{B} = 180 - 53.1... - 30\ = \underline{96.9^\circ}$$

(c) Using cosine rule,



$$OB^2 = 5^2 + 6^2 - 2\times 5\times 6\times \cos 135^\circ$$

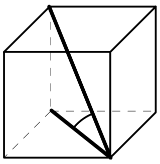
so OB = $\underline{\text{10.2 miles}}$ (to 3 s.f.)

Now using the sine rule,

$$\frac{\sin A\hat{O}B}{6} = \frac{\sin 135^\circ}{10.16...},\ \text{which gives}\ A\hat{O}B = 24.7^\circ$$

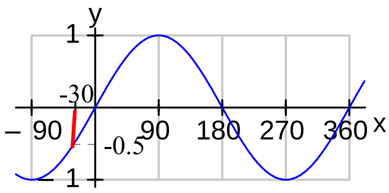
The bearing of B from O is therefore $\underline{069.7^\circ}$

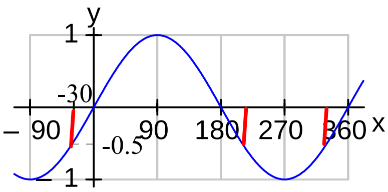
(d) Assume the length of side of the cube is 1. (Enlargement won't make any difference to the angles). Pythagoras gives the projection on the base as $\sqrt{2}$, and the opposite side is 1.



So $\tan\theta = \dfrac{1}{\sqrt{2}}$. an d $\theta = \underline{35.3^\circ}$ to 3 s.f.

(e) First find the principal value from the calculator: -30°. Where are there other angles in our window with the same sine?





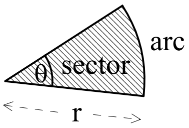
Clearly at 30° beyond 180° and 30° back from 360°.
So $x = \underline{210^\circ, 330^\circ}$

15. Circles

top

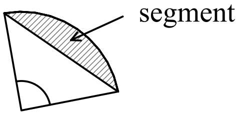
(a) arcs, sectors, segments

$$\boxed{\text{Arc length} = \frac{\theta}{360} \times 2\pi r}$$



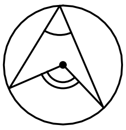
$$\boxed{\text{Sector area} = \frac{\theta}{360} \times \pi r^2}$$

Segment area = Sector – Triangle

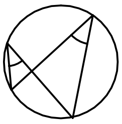


(b) circle theorems

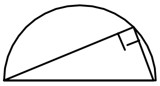
(i) Angle subtended at the centre = 2× angle subtended at the circumference by the same arc



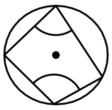
(ii) Angles in the same segment are equal



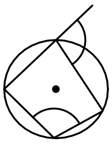
(iii) Angle in a semicircle = 90º



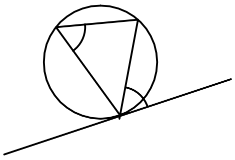
(iv) Opposite angles of a cyclic quadrilateral add up to 180º



(v) Exterior angle of a cyclic quadrilateral = interior opposite



(vi) Alternate segment theorem: the angle between tangent and side = interior opposite



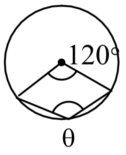
Questions

(a) The arc of a sector of a circle of radius 20cm has length 10cm.
Find the area of the sector.

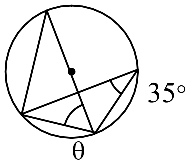
(b) A cylindrical tank, radius 50cm and length 2m with horizontal axis is partially filled with oil to a maximum depth of 25cm. How much oil is contained in the cylinder?

(c) Find θ in the following diagrams:

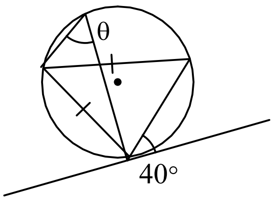
(a)



(b)



(c)

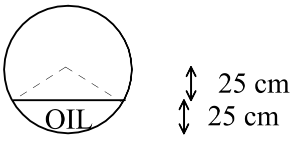


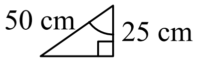
Answers

(a) Arc length = $\frac{\theta}{360}\times 2\pi 20$ and this is given as 10cm. Rearranging gives $\theta = \frac{90}{\pi}$. Therefore sector area = $\frac{\theta}{360}\times\pi 20^2 = \frac{90}{360\pi}\times\pi 20^2$ which simplifies nicely to 100cm^2.

{Would you have reached for the calculator at $\theta = \frac{90}{\pi}$*, and missed the beautiful cancellation later?}*

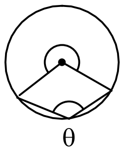
(b) We need to find the area of the segment comprising the cross-section of the oil. Above the oil is an isosceles triangle, so split it down the line of symmetry:



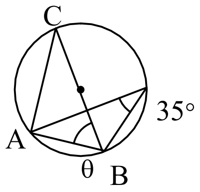


This gives an angle of $\cos^{-1} 0.5 = 60^{\circ}$, and a base of 43.3…cm by Pythagoras. So the angle at the centre of the sector is 120°. Therefore the area of the segment is $\frac{120}{360}\times\pi 50^2 - 43.3...\times 25 = 1535\text{cm}^2$. The oil is in the shape of a prism with volume $1535\times 200 = 307092$ cm^3 = <u>0.307m^3</u>.

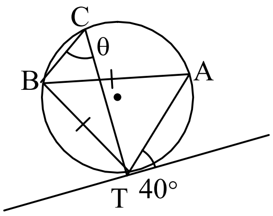
(c) The angle subtended at the centre is 360º – 120º = 240º, so $\underline{\theta = 120^{\circ}}$ by the angle at the centre theorem.



$\hat{C} = 35^{\circ}$ (angles in the same segment)
$C\hat{A}B = 90^{\circ}$ (angle in a semicircle)
so $\theta = 180 - 35 - 90 = 55^{\circ}$ (angle sum of a triangle)



$A\hat{B}T = 40^{\circ}$ (alternate segment theorem)
Isosceles triangle gives $B\hat{A}T = 70^{\circ}$, and so $\underline{\hat{C} = 70^{\circ}}$ (angles in same segment)



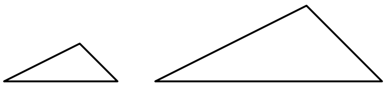
16. Similar triangles, congruent triangles

top

(a) Similar triangles

same shape, different size; related by enlargement (may be different orientation)

to prove similar: **AAA**
(each pair of angles equal)



to solve problems use either (a) scale factor or (b) ratio of sides equal

(b) Congruent triangles

same shape and size, i.e. identical though usually in different positions.

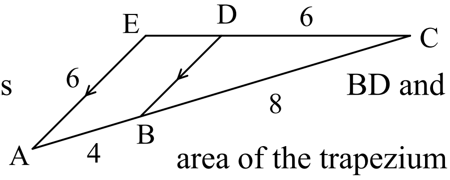
to prove congruent:

SSS
SAS
AAS
ASA
RHS

but not ASS – there are sometimes two different triangles with the same ASS

Questions

(a) (i) Prove that triangles BCD and ACE are similar. (ii) Hence find the lengths BD and DE. (iii)If the area of triangle BCD is 12 what is the area of the trapezium ABDE?



(b) Use congruent triangles to prove that the diagonals of a parallelogram bisect each other.

Answers

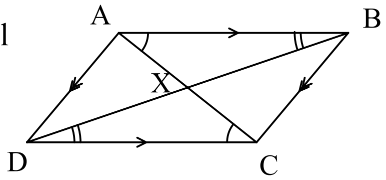
(a) (i) $E\hat{A}C = D\hat{B}C$ and $A\hat{E}C = B\hat{D}C$ (corresponding). The third angle is shared, so AAA is established and they are similar.

(ii) scale factor of enlargement is $\frac{12}{8} = \frac{3}{2}$. So BD = $6 \div \frac{3}{2}$, or

$6 \times \frac{2}{3}$ = 4. CE is $6 \times \frac{3}{2} = 9$, so DE is 9 – 6 = 3

(iii) Area of triangle ACE = $12 \times (\frac{3}{2})^2$ *{note area scale factor = k^2}*

= 27. So the trapezium has area 27 – 12 = 15.



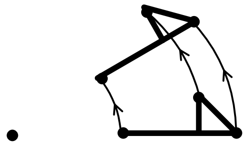
(b) The two pairs of marked angles are equal (alternate), and the top and bottom sides are equal (parallelogram). So we have two congruent triangles ABX and DCX by ASA. (Note each would have to be rotated 180º about X to transform onto the other).
So AX = XC and DX = XB, i.e. the diagonals bisect each other.

17. **Transformations**

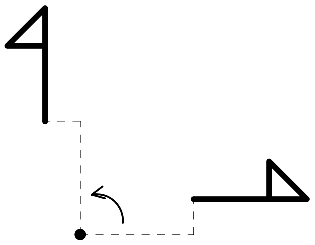
top

(i) translation by vector $\begin{pmatrix} a \\ b \end{pmatrix}$ shifts *a* to the right and *b* up.

(ii) rotation about P through θ. [Note e.g. +90º means 90º anticlockwise]
perform a rotation using compasses,



or if a multiple of 90º, use the L shape:

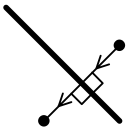


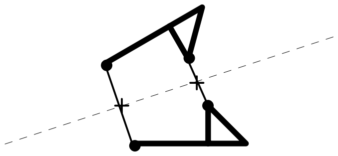
To find the centre of a rotation already performed, perpendicularly bisect a line joining any point with its image. Repeat with another pair, then where the two perpendicular bisectors meet is the centre of rotation.

Alternatively, if it's a 90º rotation, find the centre by trial and error then confirm by using L shapes.

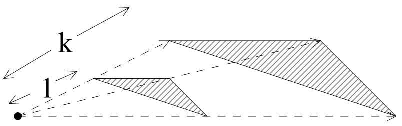
(iii) reflection through a line *l*.

To find the mirror line of a given reflection, join a point to its image and mark the mid-point . Repeat this with another pair of points, and join the two mid-points to form the mirror line.





(iv) enlargement from P with a scale factor *k*.



Note distance from centre of enlargement is multiplied by *k*.

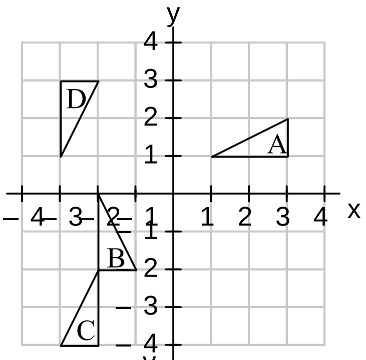
To find a centre of enlargement, join a point to its image and extend the line back. Repeat, and the centre is where the lines intersect.

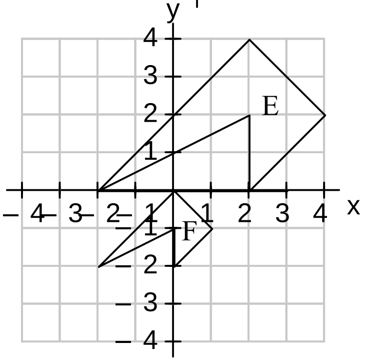
Questions

(a) What single transformation will carry triangle A onto (i) B (ii) C?

(b) A “glide reflection” is a reflection followed by a translation. A is transformed onto D by a glide reflection, in which the mirror line is $y = x - 1$. What is the vector of the subsequent translation?

(c) E is transformed onto F: state the single transformation which accomplishes this.



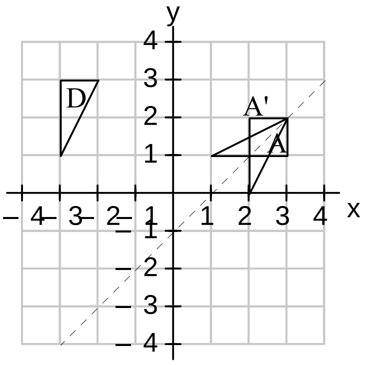


Answers

(a) (i) -90º rotation about (-1,2). *(Check with L shapes)*

(ii) reflection through the line $y = -x - 1$

(b) The diagram shows A reflected to A’. The vector of translation necessary to take A’ onto D is $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$.



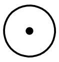
(c) Draw lines joining points with their images, and extend them downwards. They all meet at the centre of enlargement. So it’s an enlargement, centre (-2, -4) with scale factor ½.

18. **Loci and ruler and compass constructions**

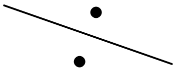
top

(a) In 2-D: locus of points equidistant from:

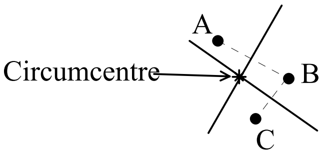
1 fixed point is a circle



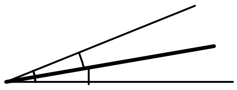
2 fixed points A and B is the perpendicular bisector of AB



3 fixed points A, B and C the circumcentre of ABC

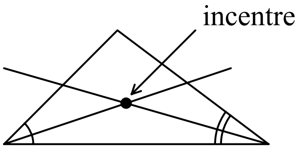


(b) locus of points equidistant from:



2 fixed lines is the angle bisector

3 fixed lines is the incentre of the triangle



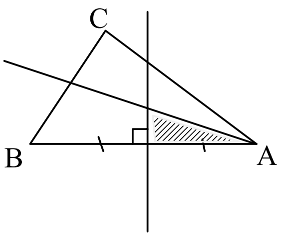
Questions

(a) Construct the triangle ABC where AB = 8cm, BC = 5cm and CA = 6cm. Construct the region of points within ABC which are closer to AB than AC, and also closer to A than B.

(b) In 3-D, describe the locus of points exactly 1cm away from the nearest point on a line segment AB.

Answers

(a) Using compasses, construct ABC accurately. Then note that the boundary lines for the two requirements are the angle bisector of $B\hat{A}C$ and the perpendicular bisector of AB, and the intersection of the two regions must be selected.



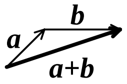
(b) This is a cylinder of radius 1cm and axis AB, and also two hemispheres of radius 1cm and centres A and B. *(or a hollow sausage, as we say in the trade).*

19. **Vectors**

top

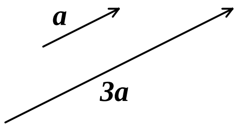
Vectors are most easily seen as journeys for a particular distance in a particular direction. location is not relevant, e.g. the opposite sides of a parallelogram can both be represented by $\underline{a}$.

$\boxed{\underline{a}+\underline{b}}$: join the arrows nose to tail:
{Note in books and exam papers vectors will be bold lower case letters without bars. *<u>You</u> write bars underneath –okay?}*



$\boxed{-\underline{a}}$: is $\underline{a}$ reversed

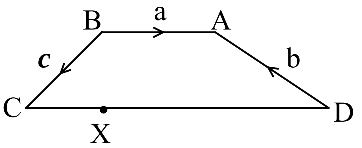
$\boxed{k\underline{a}}$ where k is a scalar is a vector parallel to $\underline{a}$, k times as long.



To get from A to B via given vectors, the route chosen doesn't matter - the expressions will all simplify down to the same answer.

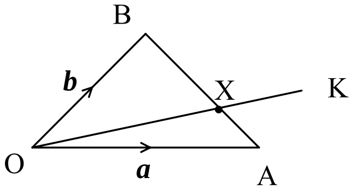
Questions

(a) ABCD is a trapezium with AB parallel to CD and CD = 2AB. The vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$ are defined as shown. X is a point ¼ of the way along CD.



(i) Find two different expressions for $\overrightarrow{AX}$ in terms of $\underline{a}$, $\underline{b}$, and $\underline{c}$.
(ii) Are the previous expressions really different? Explain.

(b) In triangle OAB, $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$.
The line OK strikes AB one third of the way up, and OK is 1½ times as long as OX.
Find in terms of $\underline{a}$ and $\underline{b}$:



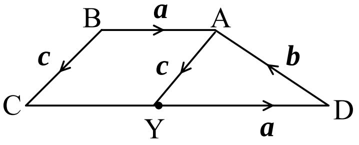
(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{OX}$ (iii) $\overrightarrow{OK}$ (iv) $\overrightarrow{AK}$
What does the final answer tell you geometrically?

Answers

(a) (i) Going via B and C we get $\overrightarrow{AX} = -\underline{a} + \underline{c} + \frac{1}{4}(2\underline{a}) = -\frac{1}{2}\underline{a} + \underline{c}$

Via D, however, we get $\overrightarrow{AX} = -\underline{b} - \frac{3}{4}(2\underline{a}) = -\underline{b} - \frac{3}{2}\underline{a}$.

(ii) These two expressions must be the same. So $-\frac{1}{2}\underline{a} + \underline{c} = -\underline{b} - \frac{3}{2}\underline{a}$, which simplifies to $\underline{a} + \underline{b} + \underline{c} = \underline{0}$. This means that each can be expressed in terms of the others, so one is superfluous. This relationship can be seen easily if we join A to the mid-point of CD and observe that there is a closed triangle illustrating that $\underline{a} + \underline{b} + \underline{c} = \underline{0}$:



(b) (i) $\overrightarrow{AB} = -\underline{a} + \underline{b}$ (ii) $\overrightarrow{OX} = \underline{a} + \frac{1}{3}(-\underline{a} + \underline{b})$, which simplifies to

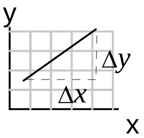
$\frac{2}{3}\underline{a} + \frac{1}{3}\underline{b}$ (iii) $\overrightarrow{OK} = \frac{3}{2}(\frac{2}{3}\underline{a} + \frac{1}{3}\underline{b}) = \underline{a} + \frac{1}{2}\underline{b}$. (iv)

$\overrightarrow{AK} = -\underline{a} + (\underline{a} + \frac{1}{2}\underline{b}) = \frac{1}{2}\underline{b}$.
That AK is parallel to OB and half as long.

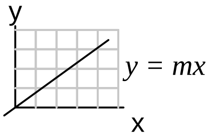
20. **Straight line graphs**

top

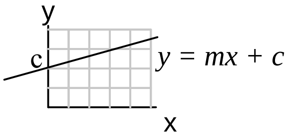
gradient $\boxed{m = \dfrac{\Delta y}{\Delta x}}$



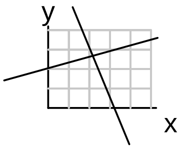
Equation of a straight line through the origin, gradient *m*, is $\boxed{y = mx}$



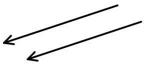
Equation of a straight line gradient *m* and y-intercept *c* is $\boxed{y = mx + c}$



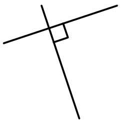
Equation of a straight line gradient m and passing through (x_1,y_1) is $\boxed{y - y_1 = m(x - x_1)}$



Intersecting lines: solve their equations simultaneously to find the intersection.



Parallel lines have $\boxed{m_1 = m_2}$

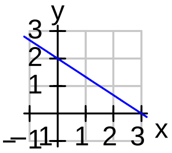


Perpendicular lines have $\boxed{m_1 m_2 = -1}$, or $m_2 = -\dfrac{1}{m_1}$

Questions

(a) What is the gradient, and y- intercept, of $2x + 6y + 12 = 0$?

(b) Find the equation of this line in the form $ax + by = c$ where the coefficients are integers.



(c) Where do the lines $y = 3x - 5$ and $3x + 2y = 6$ intersect?

(d) A is (2,3), B is (5,6) and C is (4,0). Find the equation of the line through C perpendicular to AB.

<u>Answers</u>

(a) Rearrange: $2x + 6y + 12 = 0$ [-2*x*, -12]

$6y = -2x - 12$ $[\div 6]$

$y = -\tfrac{1}{3}x - 2$

so the gradient is $-\tfrac{1}{3}$ and the y-intercept is –2.

(b) gradient = $-\tfrac{2}{3}$ and y-intercept = 2.

The equation is $y = -\tfrac{2}{3}x + 2$ $[+\tfrac{2}{3}x]$

$\tfrac{2}{3}x + y = 2$ $[\times 3]$

$\underline{2x + 3y = 6}$

(c) Solve simultaneously. In this form, substitution would be easier:

sub (1) into (2): $3x + 2(3x - 5) = 6$ which gives $x = \dfrac{16}{9}$. Sub back into (1) gives $y = \dfrac{1}{3}$. So the intersection is $\underline{(\dfrac{16}{9}, \dfrac{1}{3})}$.

(d) gradient AB is $\dfrac{6-3}{5-2} = 1$. Perpendicular gradient is $-\dfrac{1}{1} = -1$.

So the required equation is $y = -x + c$ but what is *c*? Get this by substituting the coordinates of a point on the line, i.e. C.

$\rightarrow 0 = -4 + c$, giving $c = 4$, and the equation is $\underline{y = -x + 4}$

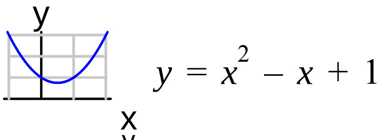
21. More graphs

top

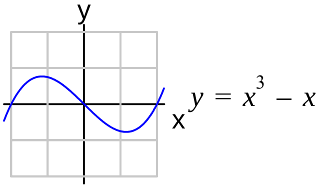
graphs of $x, x^2, x^3, \frac{1}{x}, k^x$ and $x^2 + y^2 = r^2$

(a) $\boxed{x}$ as above, linear

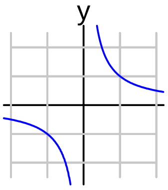
(b) $\boxed{x^2}$ parabolae



(c) $\boxed{x^3}$ cubics(!)

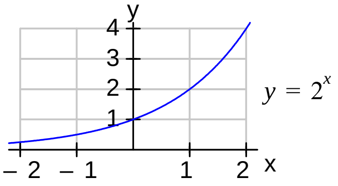


(d) $\boxed{\frac{1}{x}}$ hyperbolae

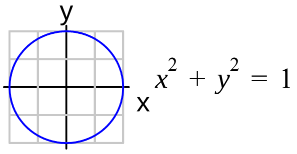


x $y = \frac{2}{x}$

(e) $\boxed{k^x}$, where $k > 0$ and x is an integer

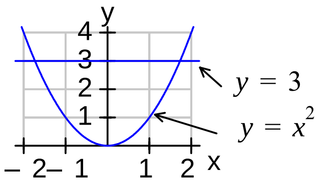


(f) $\boxed{x^2 + y^2 = r^2}$ circle radius r, centre origin



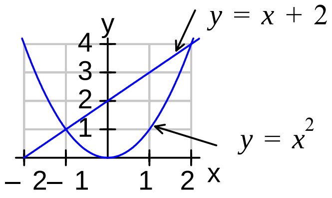
Solving equations using graphs:

(i) Draw the graph of $y = x^2$ and on the same grid $y = 3$. What equation do the intersections solve? Solving simultaneously, $x^2 = 3$, i.e. the x-values at the intersections are solutions of $x^2 = 3$, i.e.



they are $\pm\sqrt{3}$.

If $y = x + 2$ is drawn, the x-values at the intersections are solutions to $x^2 = x + 2$, i.e. $x^2 - x - 2 = 0$, which could be factorised to $(x-2)(x+1) = 0$, giving $x = -1, 2$, which can be seen on the graph.

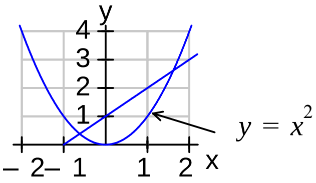


(ii) What other graph should be drawn on the same grid as $y = x^2$ to see the solutions of $x^2 + 3x - 1 = 0$? Unravel this to $x^2 = -3x + 1$ and so we need to draw the line $y = -3x + 1$.

Questions

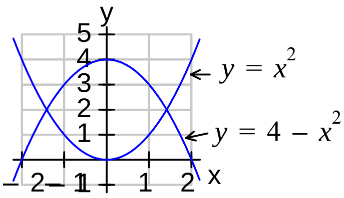
(a) Plot $y = x^2$ and $y = 4 - x^2$ on the same grid and find the *x*-values of their intersections. To what equation are these the solutions?

(b) What are the *x*-coordinates at the intersections of these two graphs? What equation is being solved approximately by these two numbers?



(c) A colony of bacteria double in number daily, after starting with 100 individuals. State the number of bacteria after (i) 1 day (ii) 2 days (ii) 3 days (iv) 4 days (v) *x* days Sketch the graph of the number of bacteria against *x*, the number of days after the start, for $0 \le x \le 5$. Estimate (a) when the colony has grown to 2500 (b) the rate of growth when *x* = 3.

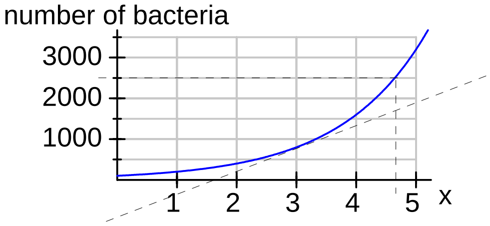
Answers



(a) Intersection *x*-values are approx. –1.4 and 1.4. At intersection, $y = x^2$ and $y = 4 - x^2$. Solving simultaneously, $x^2 = 4 - x^2$, $\therefore 2x^2 = 4$, so the equation is $x^2 = 2$. (which means these two values of *x* are actually $\pm\sqrt{2}$)

(b) At the two intersections, *x* = -0.6 and *x* = 1.6 (approx). For the line, *m* = 1 and *c* = 1, so its equation is $y = x + 1$. Therefore the equation representing *x*-values at intersection is $x^2 = x + 1$, i.e. $x^2 - x - 1 = 0$.

(c) (i) 200 (ii) 400 (iii) 800 (iv) 1600 (v) 100×2^x



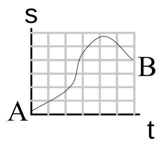
(a) 2500 are attained after about 4.7 days, (b) the gradient of the tangent at *x* = 3 shows the rate of growth at that moment, and is about 550 bacteria/day.

22. Distance, velocity graphs

top

{We are really dealing with displacement, *i.e. how far along a certain route, usually a straight line, from an origin. e.g. going round a complete circle would represent 0 displacement, but $2\pi r$ of distance}*

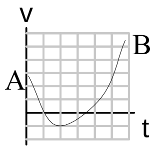
(a) **Displacement – time**



Gradient measured between A and B
= average velocity

Gradient of a tangent
= velocity at that point.

(b) **Velocity – time**



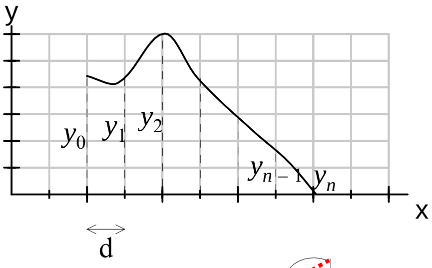
Gradient measured between A and B
= average acceleration

Gradient of a tangent
= acceleration at that point

Area between the curve and *x*-axis = displacement
{note: under the x-axis, area counts negative}

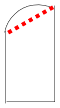
(c) **Trapezium rule**

an approximate method for counting area under a curve:



$$\text{Area} \approx \frac{d}{2}\{y_0 + 2y_1 + ...+ 2y_{n-1} + y_n\}.$$

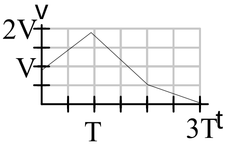
This replaces each strip with a trapezium, i.e. the top becomes a straight line segment, and will under- or over- estimate the true area.



Questions

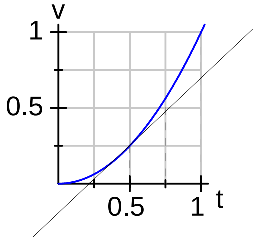
(a) An object moves in a straight line so that its velocity after time *t* seconds is given by $v = t^2$. Find
(i) its average acceleration over the first second
(ii) its instantaneous acceleration at $t = \frac{1}{2}$
(iii) the distance it covers using the trapezium rule with 4 strips.

(b) In the journey represented in the diagram, the total distance covered was 60m, and the acceleration over the first part was 5 ms^{-2}. Find the values of V and T.



Answers

(a) (i) average acceleration = change in velocity over time taken = 1 m/s per s = 1ms^{-2}.
(ii) acceleration at t = ½ is gradient of tangent there, i.e. 1ms^{-2}.
(iii) Using trapezium rule, distance



$$\approx \frac{0.25}{2}\left\{0 + 2\times 0.25^2 + 2\times 0.5^2 + 2\times 0.75^2 + 1^2\right\} = 0.34375, \text{ or}$$

0.34 m to 2 s.f. *{Note that 0.34375 is an overestimate due to the concave curve}*

(b) Splitting into two trapezia and a triangle, area under curve

$$= \frac{1}{2}(V + 2V)T + \frac{1}{2}(2V + \tfrac{1}{2}V)T + \frac{1}{2}T\frac{V}{2} \text{ which} = 3VT \text{ . So } 3VT = 60$$

Acceleration on first part = $\frac{V}{T}$ which = 5. Substituting gives

$5T^2 = 20$ which leads to $T = 2$, and $V = 10$.

23. Sequences; trial and improvement

top

(a) Sequences

Numbers in a sequence *u* receive the names $u_1, u_2, u_3, ...u_n, ...$
A sequence may be defined directly: $u_n = 3n + 1$ (that is 4, 7, 10, ….)
or recursively: $u_n = 2u_{n-1} - 3$ and $u_1 = 4$ (that is 4, 5, 7, 11, 19, 35,…)

special sequences:

(i) Triangle numbers	1, 3, 6, 10, 15, 21, 28, 35, …… where $u_n = \frac{1}{2}n(n+1)$
(ii) Fibonacci sequence	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ….. where $u_1 = 1, u_2 = 1$, and $u_n = u_{n-1} + u_{n-2}$

(b) trial and improvement

to solve an equation or maximise/minimise a quantity
e.g. Solve $x^3 - x - 1 = 0$ correct to 1.d.p.

x	$x^3 - x - 1$
0	-
1	-
2	+
1.5	+
1.3	-
1.4	+
1.35	+

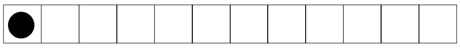
We've established there is a zero between 1.3 and 1.4, but which figure do we quote? Must go halfway, i.e. 1.35 to indicate. Answer is between 1.3 and 1.35, so when rounded it will definitely be $\underline{x = 1.3}$

<u>Questions</u>

(a) $u_n = 3n - 7$. What is (i) the 10th term (ii) the first term over 1000?

(b) Suggest a formula for –5, 2, 9, 14, 21, 28, ….

(c) Find the number of straight lines joining *n* dots, and prove your formula.



(d) In a game, a counter can move either one or two spaces on each turn. How many different ways are there for the counter to get from the 1st square to the 10th square?

(e) Find, to 1 d.p. the value of *x* which minimises the function $x^2 + 2^x$

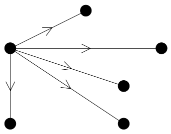
<u>Answers</u>

(a) (i) $u_{10} = 3\times 10 - 7 = 23$.

(ii) Need $u_n > 1000$, i.e. $3n - 7 > 1000$ which solves to $n > 335\frac{2}{3}$

so the first term is the 336th and is $\underline{u_{336} = 1001}$

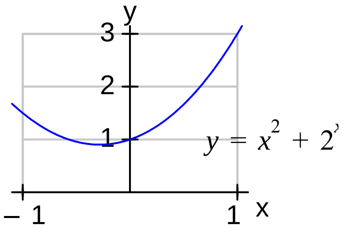
(b) $u_n = 7n - 12$, *(or, not so impressive,* $u_n = u_{n-1} + 7$ *and* $u_1 = -5$*)*



(c) Each dot radiates *n* -1 lines to the other dots, and as there are *n* dots to radiate lines from, there are $n(n-1)$ lines: except that we have counted every line exactly twice over.
So the number of lines between *n* dots is $\frac{1}{2}n(n-1)$

(d) We are having to advance the counter 9 places. Let the number of ways of advancing it *n* places be called u_n, (and we need to find u_9.)
The first move is either a 1 or a 2, after which the number of ways remaining to get to the end is u_{n-1} or u_{n-2} respectively. So $u_n = u_{n-1} + u_{n-2}$ and the sequence is our old friend the Fibonacci. Noting that $u_1 = 1$ and $u_2 = 2$, the sequence must go 1, 2, 3, 5, 8, 13, 21, 34, 55, …. and u_9 is <u>55</u>

(e) To get an idea where to look see sketch:
The minimum is around x = -0.3



x	$x^2 + 2^x$
-0.5	0.957..
-0.4	0.917..
-0.3	**0.902..**
-0.2	0.910..

So far, we are assuming there is a simple minimum, but all we know is that it's somewhere in the vicinity of –0.3 – it may well not be closest to that value at 1 d.p., so we need to go to a finer division:

x	$x^2 + 2^x$
-0.29	0.902..
-0.28	0.901..
-0.27	0.902..

We now know it's between –0.27 and –0.29, so rounded to 1 d.p. the value of x is indeed –0.3
{Provided the function is a straightforward one with no funny business}

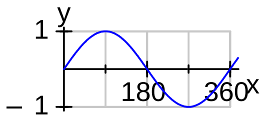
24. Graphical transformations

top

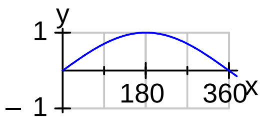
For any graph $y = f(x)$,

$\boxed{y = f(x-a)}$	$\rightarrow +a$	a translation of *a* steps in the + *x* direction
$\boxed{y - a = f(x)}$ {i.e. $y = f(x) + a$ }	$\uparrow +a$	a translation of *a* steps in the + *y* direction
$\boxed{y = f(\frac{x}{a})}$	$\leftrightarrow \times a$	a stretch by factor a in the + *x* direction
$\boxed{\frac{y}{a} = f(x)}$ {i.e. $y = af(x)$ }	$\updownarrow \times a$	a stretch by factor a in the + *y* direction

e.g.

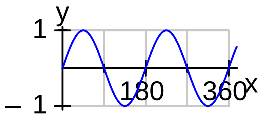


$y = \sin x$



$y = \sin\frac{x}{2}$

stretch by 2 in x



$y = \sin 2x$

stretch by 1/2 in x

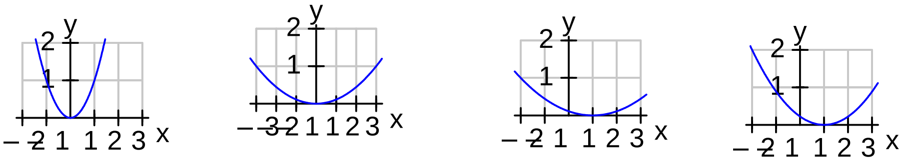
$\boxed{\text{multiple transformations:}}$

this is truly tricky. It's different from techniques in rearranging formulae: you must do a sequence of steps in the formula, each time replacing *x* (or *y*) with something new, eventually to get the required equation

e.g. what transformation must be performed on the curve $y = x^2$ to obtain the following: (a) $y = (x-1)^2 + 3$ (b) $y = 2(\frac{x-1}{3})^2$

(a) starting with $y = x^2$, replace *x* by $x - 1$ {$\rightarrow y = (x-1)^2$ }, then replace *y* by $y - 3$ {$\rightarrow y - 3 = (x-1)^2$ }. So the original parabola must be moved 1 step to the right then 3 steps up.

(b) Instinct says replace x by x-1 first, but it don't work! Starting with $y = x^2$ Replace x by $\frac{x}{3}$ $\{\rightarrow y = (\frac{x}{3})^2\}$ then replace x by $x-1$ $\{\rightarrow y = (\frac{x-1}{3})^2\}$ and finally replace y by $\frac{y}{2}$ $\{\rightarrow \frac{y}{2} = (\frac{x-1}{3})^2\}$. Thus the transformations are: a stretch in the x direction by factor 3, then a translation by +1 in the x direction, and finally a stretch by factor 2 in the y direction, illustrated here:

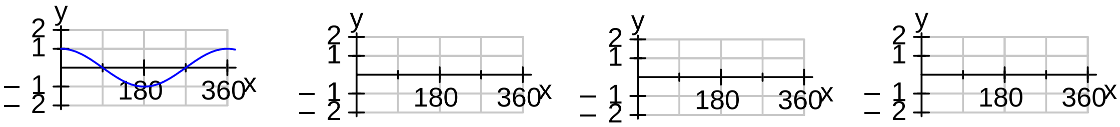


$$y = x^2 \longrightarrow y = \left(\frac{x}{3}\right)^2 \longrightarrow y = \left(\frac{x-1}{3}\right)^2 \longrightarrow y = 2\left(\frac{x-1}{3}\right)^2$$

Questions

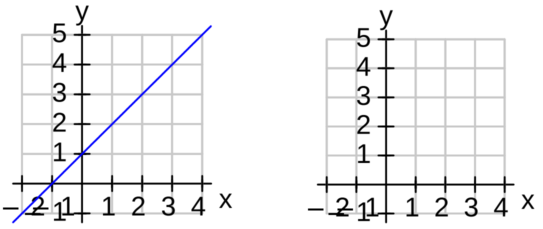
(a) The graph of $y = \cos x$ is shown. On the empty grids, sketch the graphs of

(i) $y = \cos 2x$ (ii) $y = 2\cos x$ (iii) $y = \cos\frac{x}{2} - 1$



$y = \cos 2x$ $y = 2\cos x$ $y = \cos\frac{x}{2} - 1$

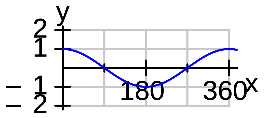
(b) On the left grid is the graph of $y = x + 1$. (i) Perform a stretch on this by a factor of 2 in the y direction, drawing the result on the empty grid. (ii) Show algebraically what effect this stretch has on the equation of the line

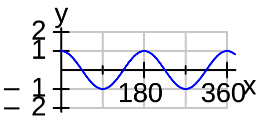


(c) Describe the transformations of the curve $y = x^2$ which result in a curve with equation $y = \frac{1}{2}\left(\frac{x}{3} + 1\right)^2$

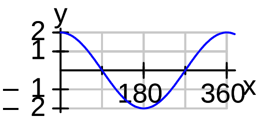
Answers

(a)

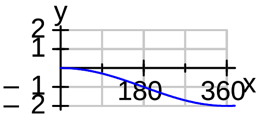




$y = \cos 2x$

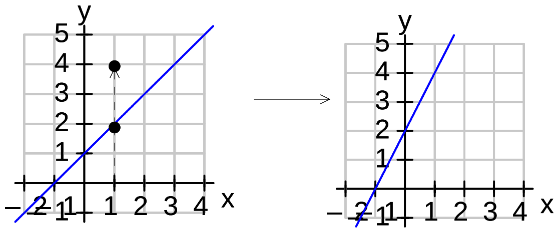


$y = 2\cos x$



$y = \cos\frac{x}{2} - 1$

(b) How do you stretch? Pick a point, measure its distance from the invariant *x*-axis, then double it.



The ensuing line will have equation $\frac{y}{2} = x + 1$, i.e. $y = 2x + 2$, and this is confirmed by the diagram.

(c) a fiendish trap. Suppress the urge to divide *x* by 3 first (as you would do in a calculation):

replace *x* by *x* +1: $\rightarrow y = (x+1)^2$. Next, replace *x* by $\frac{x}{3}$: $\rightarrow y = (\frac{x}{3}+1)^2$,

finally replace *y* by 2*y*: $\rightarrow 2y = (\frac{x}{3}+1)^2$ which is it. So the transformations are:

translate by –1 in *x* direction, then stretch by factor 3 in the *x* direction, then stretch by factor ½ in the *y* direction.

25. Probability

top

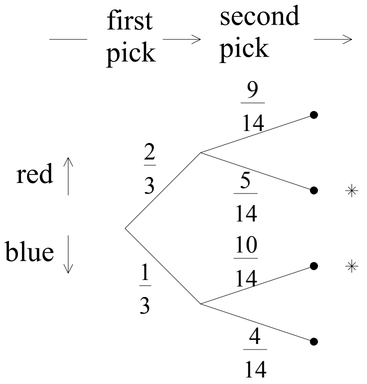
One definition of the probability of an event is the limit to which the relative frequency (no. of successes ÷ no. of trials) tends as the no. of trials $\rightarrow \infty$.

So 7 tails from 10 flips of a fair coin gives 0.7 relative frequency, as does 700 tails out of 1000 flips. However, one would expect by $n = 1000$ to have converged more closely to 0.5. That would cast doubt on the fairness of the coin.

(i) For mutually exclusive events A and B, $\boxed{P(A\,or\,B) = p(A) + P(B)}$

(ii) For independent events A and B, $\boxed{P(A\,and\,B) = P(A) \times P(B)}$

(iii) Probability trees can illustrate combined events well:
e.g. a bag contains 5 blue balls and 10 red balls. Pick one ball at random, keep it out, then pick another ball. What is the probability of one of each colour?



The two nodes corresponding to one of each colour are marked.
The probability is $\frac{2}{3} \times \frac{5}{14} + \frac{1}{3} \times \frac{10}{14}$ which is $\frac{10}{21}$.

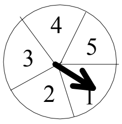
Alternatively to a tree, just spell out the sequence of events:
P(one of each colour) = P(R_1 and B_2 or B_1 and R_2) and use the addition and multiplication laws to get the same result.

Questions

(a) Draw a table of results for the rolling of two dice.
What is the probability that (i) the difference is 2 (ii) the total is 6
(iii) the difference is 2 or the total is 6 ?

(b) A teacher picks 2 pupils at random to be class representatives out of a class with 10 boys and 12 girls. What is the probability that
(i) they are both boys (ii) there is at least 1 girl ?

(c) A game consists of three turns of an arrow which lands randomly between 1 and 5, with the scores are added together.
A prize is given for a final score of 14 or more



What is the probability that (i) a player scores the same number on each turn (ii) a player wins a prize ?

Answers

(a) The difference being 2 is shown with dots while the total being 6 is shown with rings.

A →

B ↓	1	2	3	4	5	6
1			•		○	
2				⊙		
3	•		○		•	
4		⊙				•
5	○		•			
6				•		

(i) P(difference = 2) = $\frac{8}{36} = \frac{2}{9}$.

(ii) P(total = 6) = $\frac{5}{36}$

(iii) P(difference = 2 or total = 6)? Cannot use the addition law directly here because they are not exclusive. (there's an overlap). Just counting gives $= \frac{11}{36}$.

(b) (i) P(B_1 and B_2) = $\frac{10}{22} \times \frac{9}{21} = \frac{15}{77}$ (ii) P(at least 1 girl) = 1 – P(both boys) = $\frac{62}{77}$.

(c) (i) P(111 or 222 ……..) = $(\frac{1}{5})^3 + (\frac{1}{5})^3 + \ldots\ldots = \frac{1}{25}$

(ii) P(prize) = P(554 or 545 or 455 or 555) = $(\frac{1}{5})^3 + (\frac{1}{5})^3 + \ldots\ldots = \frac{4}{125}$.

26. Statistical calculations, diagrams, data collection

top

(a) calculations

(i) averages:

$$\text{mean} = \frac{\sum x_i}{n}$$

median = value of the middle item when listed in order

mode = most commonly occurring value

(ii) measures of spread:

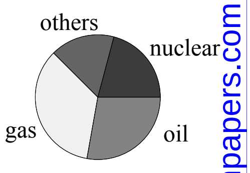
range = max – min

Interquartile range = Upper quartile – lower quartile

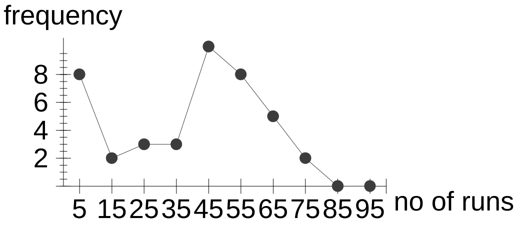
Quartiles in small data sets: fiddly and pointless, but here we go. Median is found. If the number of data was even, split the data into two sets; if the number of data was odd, ignore the median and consider the remaining values as two sets. Then the quartiles are the medians of the two remaining sets.

(b) diagrams

(i) pie chart — for categoric data (non-numerical) e.g. modes of transport used to school

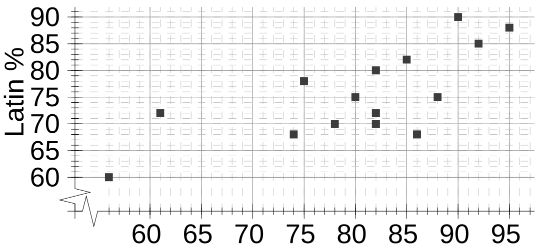


(ii) frequency diagram



(iii) moving average — in a time sequence, the mean of the last 10 (say) values is calculated then plotted. This smooths out short term fluctuations so that a long term trend may be seen.

(iv) scatter graphs — to see correlation between two variables



(v) stem and leaf diagrams the data is transcribed straight from a table onto the stems: this is a back-to-back stem and leaf.

Maths		Latin
520	9	0
8652220	8	0258
854	7	0022558
1	6	088
6	5	

key: | 3 | 6
means 36%

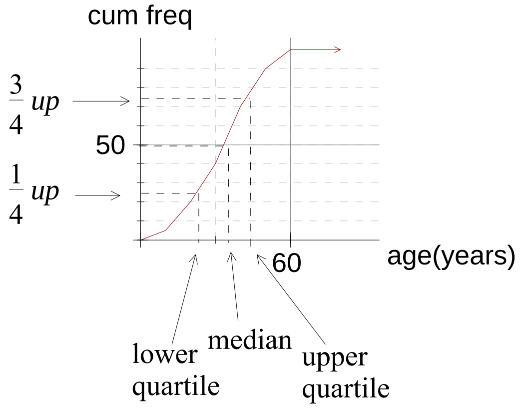
(vi) cumulative frequency curve: grouped data with frequencies is turned into cumulative frequencies thus:

x	freq.
$0<x\leq10$	5
$10<x\leq20$	8
$20<x\leq30$	12
…….	……

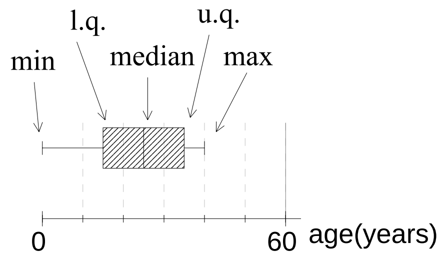
→

x	cum.freq
$0<x\leq10$	5
0$<x\leq20$	13
0$<x\leq30$	25
……	……

and the cum freq's plotted at the **right end** of the interval.



(vii) box plot



(vii) histogram: no gaps allowed. If the data is integer valued, the class boundaries will be between integers.
Height of block is not frequency, but
frequency density = freq ÷ width

(c) data collection

(i) sampling: random sampling – the population is in a numbered list, then adapt random numbers to select members repeatedly without inherent bias. e.g. a random sample of size 50 to be selected from the school population of 830: take a Ran# from calculator, $\times 1000$, discard if over 830, otherwise choose that member of the list. Repeat 50 times.

stratified sampling – when the population is divided into strata and you wish each stratum to be represented in the sample proportionately to its size. Calculate the sizes and then sample randomly within each stratum. e.g. in a prison in the age groups 18 - 25, 26 - 40, 41 – 60, and 61- 100 there are 100, 300, 250 and 150 inmates. You wish to take a stratified sample of size 50. From the 18 – 25 group you must take a random sample of size $\frac{100}{800} \times 50$, i.e. 6 (nearest integer), and so on.

(ii) questionnaires : no vague or leading questions. No questions which could have a variety of possible responses, better yes/no or tick boxes, or score on a scale of 1 to 5, say.

27. Functions

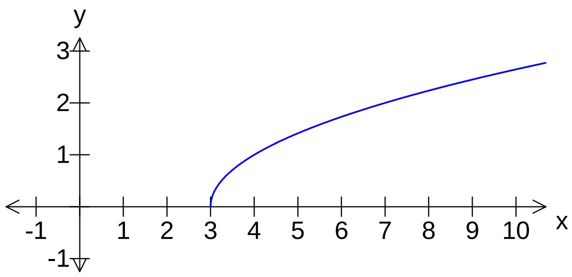
top

Functions are rules which require an input, *x*, and give a single output, $f(x)$, (also called *y*). So for example, pressing a calculator button performs a function.

Domain

This is the set of input values. This may be given in a question, or you may have to find the **natural domain**, that is the set of all possible input values.

Tha natural domain of $f(x)=\sqrt{x-3}$ is $x \geq 3$, since any values of *x* below 3 do not give a real output.



Range

This is the set of output values.

For $f(x)=\sqrt{x-3}$, the range is all the numbers between 0 and infinity, i.e. $f(x) \geq 0$ (or $y \geq 0$).

{Note that the $\sqrt{\ }$ *function only gives the positive square root}*

Composing functions

If the output from one function *f* is used as the input fro another function *g*, giving the composite function $g(f(x))$ (said as "g of f of x").

For example, if $f(x)=2x+1$ and $g(x)=3x-2$,

then $g(f(1))$ is $g(3)$ which = 7.

For this pair of functions, more generally, $g(f(x))=3(2x+1)-2$, which can be simplified to $g(f(x))=6x+1$.

Inverse

The inverse of a function, called $f^{-1}(x)$, reverses the action of the function. e.g. with $f(x)=2x-1$, $f(3)=5$, so $f^{-1}(5)$ should = 3.

To find a formula for the inverse of $f(x)$, call this *y*, and rearrange the formula so that *x* is the subject.

$$y=2x-1$$
$$\therefore y+1=2x$$
$$\therefore \frac{y+1}{2}=x$$

So $f^{-1}(y)=\frac{y+1}{2}$, but this is normally rewritten as $f^{-1}(x)=\frac{x+1}{2}$, since the input number to any function is usually called *x*. Check with the above example,

$f^{-1}(5)=\dfrac{5+1}{2}=3$, which is correct!

Questions

(a) Find the natural domains of :

(i) $f(x)=3x-2$ (ii) $f(x)=\dfrac{1}{x}$ (iii) $f(x)=\sqrt{5-x}$

(iv) $f(x)=\dfrac{1}{\sqrt{2x-2}}$ (v) $f(x)=\sqrt{x(4-x)}$

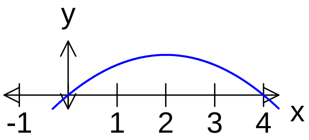
(b) Find the ranges of all the functions in q.1

(c) If $f(x)=5x-3$ and $g(x)=\dfrac{x-3}{x+2}$, find a simplified expression for $g(f(x))$.

(d) Find the inverse of (i) $f(x)=\dfrac{2x+3}{5}$ (ii) $h(x)=\dfrac{x-3}{x+1}$

Answers

(a) (i) the entire real line
(ii) all real numbers except 0
(iii) $x \le 5$
(iv) $x > 1$
(v) the inside bit needs to be ≥ 0. This means $x(4-x) \ge 0$, which is a quadratic inequality:
it requires $0 \le x \le 4$.



(b) (i) the entire real line
(ii) all real numbers except 0
(iii) $y \ge 0$
(iv) $y > 0$
(v) $0 \le y \le 2$

(c) $g(f(x)) = \dfrac{(5x-3)-3}{(5x-3)+2} \quad = \quad \dfrac{5x-6}{5x-1}$

(d) (i) $y = \dfrac{2x+3}{5}$

$$5y = 2x+3$$
$$5y-3 = 2x$$

$\dfrac{5y-3}{2} = x$, so $f^{-1}(x) = \dfrac{5x-3}{2}$.

(ii) $y = \dfrac{x-3}{x+1}$

$$(x+1)y = x-3$$
$$xy + y = x-3$$
$$xy - x = -3-y$$
$$x(y-1) = -3-y$$

$x = \dfrac{-3-y}{y-1}$. Why all these minuses? Let's multiply top and bottom by -1, and

$h^{-1}(x) = \dfrac{3+x}{1-x}$.

top

28. Calculus

Differentiation

If $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) is the name of the **gradient function** of y.

How to differentiate: If $\boxed{y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}}$

Constants multiplying a power of *x* remain:

If $y = 10x^2$ then $\frac{dy}{dx} \quad = \quad 10 \times 2x \quad = \quad 20x$

Rate of change

$\frac{dy}{dx}$ is the **rate of change** of *y* with respect to *x*. So, for example:

(i) on the graph of *y* against *x*, $\frac{dy}{dx}$ represents the **gradient**.

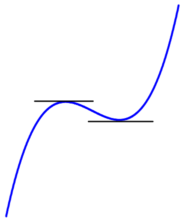
(ii) if *h* metres is the vertical height of a ball after *t* seconds, then $\frac{dh}{dt}$ is the vertical **velocity** of the ball in m/s.

(iii) if *P* is the price of a share, $\frac{dP}{dt}$ is the **rate of change** of the share price.

Max/Min

Put $\frac{dy}{dx} = 0$ and solve.

At a max or min the gradient will be 0.



Is the stationary point ($\frac{dy}{dx} = 0$**)** you have located is a max or a min? To determine this, factorise the gradient function if possible, and calculate the gradients at the stationary point, and also nudging a little to the left and a little to the right:

e.g. On $y = x^3 - 3x$ show that there is a staionary point at $x = 1$ and determine its nature.

$\frac{dy}{dx} = 3x^2 - 3$

when $x = 1$, $\frac{dy}{dx} = 3 \times 1^2 - 3 = 0$, so there <u>is</u> a stationary point.

What type is it? $\frac{dy}{dx}$ can be factorised to $\frac{dy}{dx} = 3(x-1)(x+1)$.

x	1^-	1	1^+
$\frac{dy}{dx}$	-	0	+
	╲	—	╱

The diagram shows that we have a minimum.

{For 1^-, imagine substituting a number just less than 1 for x in the expression $\frac{dy}{dx} = 3(x-1)(x+1)$, and it's clearly going to be negative, etc}

Questions

(a) Differentiate the following functions:

(i) x^3 (ii) 17.5 (iii) $5x^2$ (iv) $3x(x+1)$

(v) $\frac{2}{x}$ (vi) $\frac{x+1}{x}$

(b) Show that the curve $y = x^3 + x$ has no stationary points.

(c) Find the coordinates of the stationary points on $y = x^3 + 3x^2 - 9x + 5$, and determine whether the right hand one is a max or a min.

(d) The displacement of a toy car during the first 10 seconds after release is given by $s = t^2 - \frac{t^3}{15}$. Find (i) the car's speed after 2 seconds (ii) the maximum speed of the car.

Answers

(a) (i) $3x^2$ (ii) 0 (iii) $10x$ (iv) $f(x)=3x^2+x$ so $f'(x)=6x+1$ (v) $f(x)=2x^{-1}$ so $f'(x)=2\times -1x^{-2}$, i.e. $\frac{-2}{x^2}$.

(vi) $\frac{x+1}{x}$ needs to be simplified to $1+\frac{1}{x}$, that is $1+x^{-1}$. So $f'(x)=-1x^{-2}$, i.e. $\frac{-1}{x^2}$.

(b) $\frac{dy}{dx}=3x^2+1$. Now $3x^2$ is always at least 0, so $\frac{dy}{dx}$ is always at least 1.
So there are no stationary points.

(c) $\frac{dy}{dx}=3x^2+6x-9$. For stationary points, $3x^2+6x-9=0$. Therefore

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$\therefore\ x=-3$ or 1.

Plugging back into the original equation gives coordinates as (-3, 32) and (1,0).

Is (1,0) a max or min?

Using $\frac{dy}{dx}=3(x+3)(x-1)$,

x	1^-	1	1^+
$\frac{dy}{dx}$	-	0	+
	╲	─	╱

so we have a minimum.

(d) $v=\frac{ds}{dt}$, so $v=2t-\frac{t^2}{5}$.

(i) After 2 seconds, $v=2\times 2-\frac{2^2}{5}$ = 3.2 m/s.

(ii) To find the maximum value of *v*, we need to differentiate the expression for *v*.

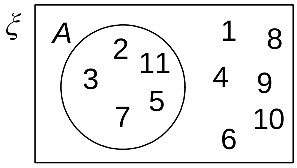
$\frac{dv}{dt}=2-\frac{2t}{5}$.

Putting this = 0 solves to $t=5$. Back into the formula for *v* gives the maximum value of *v* as $2\times 5-\frac{5^2}{5}$ = 5 m/s.

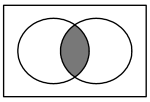
top

29. Sets

Sets are groups of elements.
A small set may be shown on a Venn diagram, e.g. given the universal set is positive integers less than 12, and A is the set of primes:

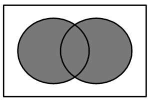


$A \cap B$:



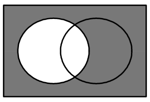
the intersection of A and B

$A \cup B$:



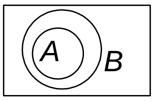
the union of A and B

A' :



the complement of A

$A \subset B$:



A is a subset of B

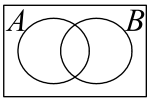
$6 \in A$: 6 is a member of the set A

$\varnothing$: the empty set

Intersections are overlaps, unions are all elements in one or the other or both.

<u>Questions</u>

(a) Shade the set $A \cap B'$:



(b) A is the set of animals, B is the set of black objects, and C is the set of cats.
(i) Translate into normal English: $B \cap C \neq \varnothing$
(ii) Describe the set $B \cap A'$
(iii) Is a white mouse a member of the set $A \cap (B \cup C)'$?

(c). In a class of 25, 12 play football, 15 play water polo, but 10 do neither sport. How many play both football and water polo?

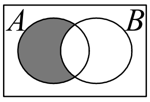
(d) ξ is the set of all employed people in England. A is the set of those with a bank account. B is the set of those with a building society account. C is the set of people who work in the catering industry.

(i) Shade the set of those in catering with a bank account but no building society account, and describe this in set notation.

(ii) Shade the set $C \cap (A \cup B)'$, and describe the members of this set.

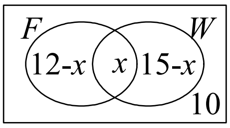
<u>Answers</u>

(a)



(b) (i) there do exist black cats
(ii) black inanimate objects
(iii) the white mouse is certainly not a member of black objects or cats, and it is an animal, so yes!

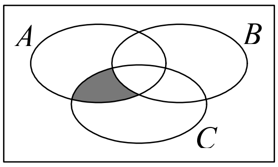
(c) let's call the number in the intersection *x*. Then the numbers in the other compartments can be calculated:



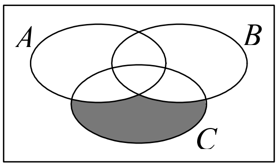
Now using the fact that they represent 25 people altogether:
$(12-x)+x+(15-x)+10=25$, so $x=12$,
i.e. there are 12 people who play both football and water polo.

(Note also that there are none in the left hand compartment, i.e. $n(F\cap W')=0$, which means that in this case $F\subset W$. That is, all who play football <u>also</u> play water polo.)

(d) (i) this is $A\cap C\cap B'$



(ii)



The members of $C\cap(A\cup B)'$ are those in the catering industry without a bank or building society account.